Faculty of Graduate Studies
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Characterizing the geometry of envy human behavior using game theory model with two types of homogeneous players

By
Reem Rajab

Supervisor
Dr. Abdelrahim S. Mousa

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Committee Members:

Dr. Abdelrahim Mousa	(Head of Committee)	....
Prof. Marwan Aloqeili	(Internal Examiner)	....
Prof. Mohammad Saleh	(Internal Examiner)	....

Declaration

To my lovely daughter and supportive husband. To my family. To my husband's family and my friends. To my Supervisor Dr. Abdelrahim Mousa for his unflagging help, advice and support. To the discussion committee Prof. Marwan Aloqeili and Prof. Mohammad Saleh for their valuable comments. To every teacher who taught me. And my apologies to those whose names not mentioned here. Thank you all from my heart.

Abstract

An envy behavioral game theoretical model with two types of homogeneous players is considered in this Thesis. The strategy space of each type of players is a discrete set with only two alternatives. The preferences of each type of players is given by a discrete utility function. All envy strategies that form Nash equilibria and the corresponding envy Nash domains for each type of players have been characterized. We use geometry to construct two-dimensional envy tilings where the horizontal axis reflects the preference for players of type one, while the vertical axis reflects the preference for the players of type two. The influence of the envy behavior parameters on the Cartesian position of the equilibria has been studied, and in each envy tiling we determine the envy Nash equilibria. We observe that there are 1024 combinatorial classes of envy tilings generated from envy chromosomes: 256 of them are being structurally stable while 768 are with bifurcation. We introduce some conditions for the disparate strategies that form envy Nash equilibria. Finally, we study the special case of the model when the envy parameters are coinciding to one.

الملخص

في هذه الرسالة نقدم نموذج الحسد لنوعين من الأفراد المتجانستين. مجموعة الفضاء لكل نوع من الأفراد هي مجموعة منفصلة تتكون من خيارين فقط. المنفعة لكل نوع من الأفراد تتمثل في إقتران منفصل. في هذا النموذج سيتم تحديد جميع استراتيجيات الحسد التي تشكل إستراتيجيات ناش المتزنة وسيتم تحديد اتزانات ناش المرافقة لكل نوع من الأفراد كما سيتم دراسة متغيرات الحسد على موقع نقاط الاتزان. سوف نستخدم الهندسة في المستوى الديكارتي لرسم مجالات الحسد المتزنة من خلال أشكال الحسد التي يعكس فيها محور (س) تأثير أفراد النوع الأول بينما يعكس محور (ص) تأثير أفراد النوع الثاني. في كل شكل من أشكال الحسد سيتم تحديد نقاط الحسد المتزنة. يوجد 1024 شكل من أشكال الحسد يتم توليدها من خلال ما يسمى كروموسومات الحسد منها 256 شكل ثابت و 768 شكل يوجد فيها اندماج. تم تحديد شروط تضمن وجود نقاط ناش المتشعبة (المتفرقة). في النهاية تم دراسة حالة خاصة من النموذج وذلك عندما تكون متغيرات الحسد متساوية للقيمة واحد.

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Introduction

Modeling the behavior of players using game theory has been studied intensively by Economists and Scientists. Ajzen [1] constructed the main goal in Planned Behavior or Reasoned Action theories to understand how players behavior were produced. In 2010, Brida et al. [3] studied the characteristics of players that might affect their decisions in a game theory model. One year later, Almeida et al. [2] studied a special game for different types of players based on the papers introduced by J. Cownley and M. Wooders [4]. In [6] Mousa et al. presented a dichotomous decision model, where players choose between two alternative decisions and can influence the decisions of the others. Soeiro et al. [11] presented new game theory model to understand how the societies' behavior could affect the market shares. In [7], Mousa and Pinto show that the pure Nash can be cohesive (all players are in favor of making same decision) or disparate (players with the same preferences are in favor of making an opposite decisions). For further readings in this context, we refer the reader to [5] and [6].

In this Thesis, we will study the influence of the envy behavior for players over the utility function of other type by extending the pure Nash equilibria studied in the game decision model [10]. We characterize all the pure envy strategies that form Nash equilibria and determine the corresponding envy Nash domains. Pure envy strategies means the cohesive or the disparate envy strategies. The strategies that are pure disparate envy Nash can explain the division of the community. For a given level of an envy behavior, we construct the corresponding geometric tiling in the Cartesian $xy-$plan, where the horizontal axis represents the relative preference of players with type t_1, and the vertical axis represents the relative preference of players with type t_2. Noting that the envy Nash domains form the decision tiles, we show that there are 1024 combinatorial classes of envy decision tilings, generated from the horizontal envy chromosomes for players of type t_1 and vertical envy chromosomes for players of type t_2, which demonstrates the high complexity of human envy behavior. Furthermore, we found 256 combinatorial classes of tilings that are being structurally stable while 768 combinatorial classes have either single or double or degenerate bifurcations. We show that the tilings connects geometrically the envy Nash equilibria.

This Thesis is organized as follows. In chapter 1 we represent a short basic review of game theory. In chapter 2 we review part of the game theory model introduced for one type of players as in [9], after that we review the decision game model with two types of players introduced by Mousa et al. in [10]. In chapter 3 we study the influence of the envy behavior for both types of players over the utility functions of each other and characterize the cohesive envy Nash equilibria, the geometric classes of pure envy Nash equilibria domains and the disparate envy Nash equilibria as presented in [8] and we study the special case of the model when the envy parameters are equal to one. Finally, we represent the Conclusions.

Chapter 1

Introduction to game theory

In this chapter, we introduce some basics of Game Theory and it's application mainly using the book reference [12].

1.1 The structure of games

In this section, we describe the essential elements of a game following the game theory book [12].

Definition 1.1. *[12] Game theory is the process of modeling the strategic interaction between two or more players in a situation that contain a set of rules and outcomes, with the right theoretical tools in place.*

Any game consists of the following formal elements:

(i) Set of players,

(ii) Strategy space for players,

(iii) A utility function or preference for players (payoff).

There are two important concepts used to define the utility function for each player. The first concept as introduced in [3] is what we called the taste type, which reflects the inner characteristics of the player that measures how much the players prefer to make a decision independently of the other. The second concept is what we called the crowding type, which reflects the influence of the other players over my decision.

1.2 Type of games

In this section, we explain the types of games in game theory, moreover various types of games help to analyze various types of problems.

Definition 1.2. *[12] Cooperative game theory assumes that groups of players, called coalitions, are the units of decision-making, they have a contractual relations and based on their contracts they make their decisions or choose the strategy they want to play and this is a cooperative behavior.*

As an example of cooperative game: a group of people together are caring out a certain project, so they cooperate between their decisions at every step.

Definition 1.3. *[12] Non-Cooperative game theory in this type of games we treat all the players' actions as separate player's action. A player action is when a person decides on his own preference, independently of the other people presented in the same game.*

Non-cooperative games refer to the games in which the players decide on their own strategy to maximize their profit. As a result, non-cooperative game theory is more common and used than cooperative games.

1.3 Representing games

The games can be represented in two forms:

- The normal (strategic) form.
- The extensive form.

1.3.1 The normal form game

Given a game G with two players $I = \{1, 2\}$. Let S_i be the strategy space of player $i \in I$. The strategy profile that describes strategies for all the players in the game is a vector $S = (s_1, s_2)$, where $s_i \in S_i$ is the strategy of player $i \in I$. Let $\mathbf{S} = S_1 \times S_2$ be the set of all strategy profiles. For each player $i \in I$, we can define the player i's payoff function as

$$U_i : S_i \longrightarrow R.$$

Definition 1.4. *[12] A game G is called normal form (or matrix form) if it consists of a set of players, $I = \{1, 2, \ldots, n\}$, strategy spaces for each player, $S_1, S_2, \ldots, S_n$ and payoff functions for the players, $U_1, U_2, \ldots, U_n$.*

Note that, all players aim to choose the strategy that maximize their utilities. A natural way to represent a two players normal form game is using a bi-matrix.

Example 1.1. *Consider a game with two players P_1 and P_2. Assume the strategy space of player P_1 is the set $S_1 = \{A, B\}$ and the strategy space of player P_2 is the set $S_2 = \{O, M\}$. The payoffs for this game is as given in the matrix form presented in Figure 1.1, Based on Figure 1.1: If Player P_1 decides the strategy B and player P_2 decides the strategy M, then P_1 gets 2 dollar and P_2 gets 1 dollar as a payoff.*

1.3.2 The extensive form

Definition 1.5. *[12] A game in the extensive form is a tree which consists of nodes and branches. Nodes represent places where something happens in the game (such as a decision by one of the players),*

P_1 \ P_2	O	M
A	1,2	0,0
B	0,0	2,1

Fig. 1.1: Normal form game.

and branches indicate the various actions that players can choose. We represent nodes by solid circles and branches by arrows connecting the nodes. And payoffs for each player located at each last node.

Definition 1.6. *Lack of information is presented in the tree by a dashed line connecting the two nodes. This means that the player knows he is at one of these nodes, but he does not know which one of them.*

If the game has no lack of information, then it is called with complete information. Otherwise, the game is called with incomplete information.

Now, we take the following extensive form example.

Example 1.2. *Consider the game in Figure 1.1. The payoff function for this game is as given in the following extensive form presented in Figure 1.2:*

The initial node belongs to player P_1, indicating that player P_1 moves first. The tree as follows: player P_1 chooses between A and B, player P_2 does not observe player P_1 choice and then chooses between O and M. The payoffs are as specified in the tree. There are four outcomes represented by the four terminal nodes of the tree: $(A, O), (A, M), (B, O)$ and (B, M). The payoffs associated with each

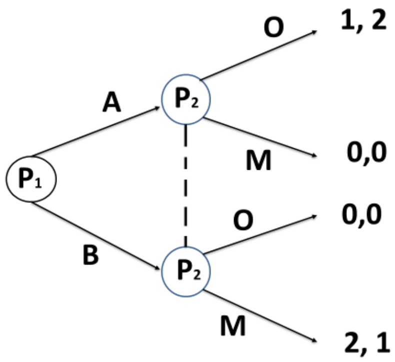


Fig. 1.2: Extensive form for a game with incomplete information.

outcome respectively are as follows $(1, 2), (0, 0), (0, 0)$ and $(2, 1)$. If the game is with complete information, then player P_2 observes the choices of player P_1. To clarify this we introduce the following example.

Example 1.3. *Consider the following extensive form*

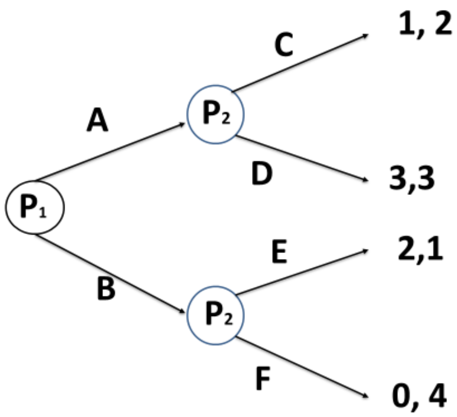


Fig. 1.3: Extensive form for a game with complete information.

- *In this game: if P_1 plays A, then P_2 observes this and plays D to maximize his payoff and so P_1 he/she gets 3.*

- *In this game: if P_1 plays B, then P_2 observes this and plays F to* *maximize his payoff and so P_1 he/she gets 0. But P_1 knows if he/she plays A, then he/she would get 3 and if he/she plays B, then he/she would get 0. Therefore, P_1 will decide A. P_1 observes the choice of P_1 and so P_2 selects the strategy that maximizes his payoff. However, P_1 knows this fact, so P_1 selects strategy that maximizes his payoff, too.*

So this game is with complete information. That is, P_1 observes the choice of P_1 and so P_2 selects the strategy that maximizes his payoff. However, P_1 knows this fact, so P_1 selects strategy that maximizes his payoff, too.

Note that, form Example 1.3 the strategy space of P_1 is $S_1 = \{A, B\}$ while the strategy space of P_2 is $S_2 = \{CE, CF, DE, DF\}$.

P_1 \ P_2	CE	CF	DE	DF
A	1,2	1,2	3,3	3,3
B	2,1	0,4	2,1	0,4

Fig. 1.4: The matrix form game of Example 1.3.

We remark that, the extensive form can be uniquely represented in matrix form. However, the converse is not true. Now, we can represent the extensive form game in Example 1.3, using the matrix form as presented in Figure 1.4 .

Note that, the matrix form of the game represented in Example 1.1, can have more than one extensive form as follows.
Note that in Figure 1.5, player P_1 starts the game while player P_2 starts the game in Figure 1.6.

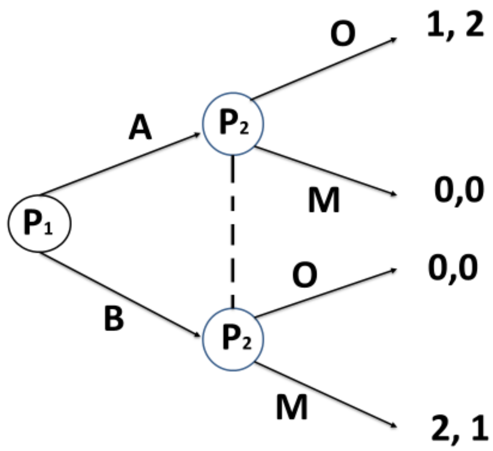


Fig. 1.5: One possible extensive form for the matrix form game presented in Example 1.1.

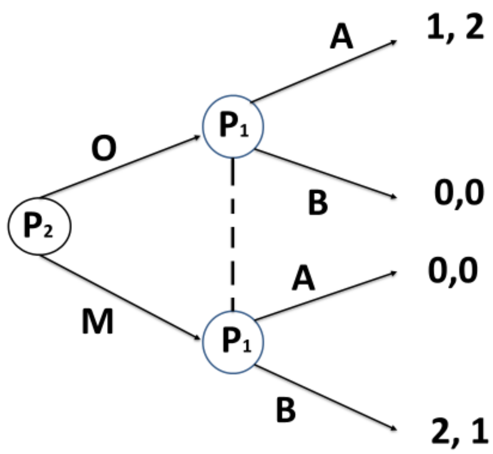


Fig. 1.6: Another possible extensive form for the matrix form game presented in Example 1.1

1.4 Dominance

Each player makes his decision in the game based on the best payoff that would be achieved. That what makes every player rational.

Definition 1.7. *[12] A pure strategy $s_i \in S_i$ of player P_i, $i \in I$, is strictly dominated if there is a strategy $\bar{s}_i \in S_i$ such that*

$$U_i(\bar{s}_i, s_{-i}) > U_i(s_i, s_{-i})$$

for all strategy $s_{-i} \in S_i$ of the other players.

Example 1.4. *Consider the following matrix form game introduced in Figure 1.7. We now study the dominance for three strategy game.*

In this game strategy N dominates strategy R for each player, since N gives payoffs higher than strategy R. So strategy R is strictly dominated by strategy N. Now, after deleting strategy R, in the new game we see that strategy N strictly dominates strategy M for each player, so strategy M is dominated by strategy N.

P_1 \ P_2	R	N	M
R	1,1	5,4	5,6
N	4,5	7,7	9,6
M	6,5	6,9	8,8

Fig. 1.7: Dominated strategies.

1.5 Nash Equilibrium

In this section, the formal definition of *Nash Equilibrium* will be introduced together with some examples.

Definition 1.8. *[12] A strategy $S^* \in \boldsymbol{S}$ is a (pure) Nash Equilibrium if*

$$U_i(S^*) \geq U_i(S), \qquad \forall i, \quad \forall S \in \boldsymbol{S} \ .$$

Example 1.5. *Consider a game between two firms F_1 and F_2. Each firm has to decide between two actions of production, A or B. The payoff is as given in the matrix form in Figure 1.8. We see that this game has no Nash Equilibrium.*

Firm F_2 can increase its payoff from 0 to 10 by adopting the action B rather than the choice A. Thus, profile (A, A) is not a Nash Equilibrium.

Firm F_1 can increase its payoff from 0 to 10 by adopting the action B rather than the choice A. Thus, profile (A, B) is not a Nash Equilibrium.

Firm F_1 can increase its payoff from 0 to 10 by adopting the action A rather than the choice B. Thus, profile (B, A) is not a Nash Equilibrium.

Firm F_2 can increase its payoff from 0 to 10 by adopting the action A rather than the choice B. Thus, profile (B, B) is not a Nash Equilibrium.

We conclude that the game has no Nash Equilibrium.

Example 1.6. *Consider the normal form game presented in Figure 1.9. We need to find the Nash equilibria strategies.*

Neither player can increase its payoff by choosing a different strategy than (C, M), so this strategy profile is a Nash Equilibrium.

Player P_1 can increase its payoff from 0 to 7 by choosing the strategy E rather than the strategy C. Thus, the profile (C, D) is not a Nash Equilibrium.

Player P_1 can increase its payoff from 0 to 5 by choosing the strategy C rather than the strategy E. Thus, the profile (E, M) is not a Nash

F_1 \ F_2	A	B
A	10,0	0,10
B	0,10	10,0

Fig. 1.8: A game with no Nash equilibrium.

Equilibrium.

Neither Player P_1 nor Player P_2 can increase their payoff by choosing different strategy than the profile (E, D), so (E, D) is a Nash Equilibrium. We conclude that the game has two Nash equilibria, (C, M) and (E, D).

P_1 \ P_2	M	D
C	5,5	0,0
E	0,0	7, 7

Fig. 1.9: A strategic game with two Nash equilibria.

Now, we find Nash equilibria for our games. In Example 1.1, the game has two Nash equilibria, (A, O) and (B, M). In Example 1.4, the game has one *Nash Equilibrium* is (N, N). In Example 1.3, the game has also one *Nash Equilibrium* (A, D).

Chapter 2

Review of a decision game model with one and two types of players

In this chapter, we review part of the game theory model introduced by Fedaa et al. [9], and part of the game decision model introduced in [10] by Mousa et al..

2.1 *Decision game model with one type of players*

The model has one type of homogeneous players $i \in \mathbf{I} = \{1, 2, \ldots, m\}$, $m \geq 2$. Each player has to make one decision $d \in \mathbf{D} = \{Y, N\}$. We define the *preference decision vector by*

$$(\omega^Y, \omega^N)$$

whose *coordinates* ω^d demonstrate how much a given player likes ($\omega^d > 0$), or dislikes ($\omega^d < 0$), or indifferent ($\omega^d = 0$) to make decision d. Note that (ω^Y, ω^N) indicates the taste type for the players (see [2, 6]).

We define the *preference neighbors vector by*

$$(\alpha^Y, \alpha^N)$$

whose *coordinates* α^d indicate how much a given player likes ($\alpha^d > 0$), or dislikes ($\alpha^d < 0$), or indifferent ($\alpha^d = 0$) to be with other players make decision d. Note that (α^Y, α^N) demonstrates the crowding type of the players and whom they are in favor or not in favor to be with in each decision (see [2, 6]).

We conclude the players' (pure) decision by

$$S : \mathbf{I} \to \mathbf{D}$$

which connects to player $i \in \mathbf{I}$ his favorite decision $S(i) \in \mathbf{D}$. Consider $\mathbf{S}$ to be the set of all possible strategies S. Given any $S \in \mathbf{S}$, we define $l = l^Y(S)$ (resp. $m - l = l^N = l^N(S)$) be the number of players that make the decision Y (resp. N). Let $\mathbf{O}$ be the occupation set which contains all possible choices of l and defined by.

$$\mathbf{O} = \{l : l \in \{0, 1, 2, \ldots, m\}\}.$$

As in [10], we now also define important parameters that play a prodigious role in classifying the equilibria.

Definition 2.1. *The difference decision parameter of the players is defined by*

$$x \;=\; \omega^Y - \omega^N. \tag{2.1}$$

Definition 2.2. *The decision parameter of the players is defined by*

$$A \;=\; \alpha^Y + \alpha^N. \tag{2.2}$$

As a consequence of Definition 2.1 if $x > 0$, then players are in favor to make a decision Y. If $x = 0$, then players are indifferent to make a decision Y or N. If $x < 0$, then players are in favor to make a decision N.

2.2 Discrete utility functions

In this section, we represent the utility function for any player $i \in \mathbf{I}$.

Let $U_i : \mathbf{D} \times \mathbf{O} \to \mathbb{R}$ be the utility function for any player $i \in \mathbf{I}$ given by

$$U_i(S(i), l) = \begin{cases} \omega^Y + \alpha^Y(l^Y - 1), & S(i) = Y \ , \\ \\ \omega^N + \alpha^N(l^N - 1), & S(i) = N \ , \end{cases} \tag{2.3}$$

For every $S \in \mathbf{S}$, the player $i \in \mathbf{I}$ has utility function of the form

$$U_i(S(i), l(S))$$

and the Nash equilibria domain is the set of all preferences x for which S is a Nash Equilibrium denoted by $N(S)$.

Following the classical definition of *Nash Equilibrium* introduced in Definition 1.8, we continue to study when the cohesive strategies are *Nash Equilibrium.*

2.3 Cohesive Nash Equilibrium strategies

In this section, we study the Nash domain intervals for all cohesive strategies $S \in \mathbf{S}$ that form Nash Equilibria. As well as, we explain how the parameters x, A and m play a significant role for characterizing the Nash equilibria.

Definition 2.3. *A cohesive strategy means all players are in favor to make the same decision.*

A cohesive strategy is described by the following map

$$C_k : \mathbf{I} \to \mathbf{D}, \quad k \in \{0, m\}$$

that demonstrates for any player $i \in \mathbf{I}$, his favorite decision. Given a *cohesive strategy* $C_k \in \mathbf{C} = \{0, m\}$, we note that cohesive strategies can have two forms :

C_m all players decide Y, i.e $m = l^y(C_m)$;

C_0 all players decide N, i.e $0 = l^y(C_0)$.

As a consequence of Definition 2.3, there are only two types of cohesive strategies: $l = m$ in which all players are in favor to make decision Y, and $l = 0$ in which all players are in favor to make decision N. The following result states the cohesive thresholds for this type of players that determine the equilibria.

Lemma 2.1. *[9] The cohesive strategy C_m is Nash Equilibrium iff*

$$x \geq -\alpha^Y(m-1)$$

and the cohesive strategy C_0 is Nash Equilibrium iff

$$x \leq \alpha^N(m-1) \ .$$

As a result of Lemma 2.1, we represent the following definition for the *cohesive threshold*.

Definition 2.4. *The cohesive threshold $C(Y)$ and $C(N)$ for the strategies Y and N are, respectively, defined by*

$$C(Y) = -\alpha^Y(m-1) \ \ and \ \ C(N) = \alpha^N(m-1) \ . \tag{2.4}$$

We now proceed to introduce the following result.

Lemma 2.2. *[9] Let $C(Y)$ and $C(N)$ be the cohesive threshold for the strategies Y and N as given in (2.4), then:*

$$C(Y) - C(N) \ = \ -A(m-1). \tag{2.5}$$

Given a *cohesive strategy* $C_K \in \mathbf{S}$, where $K \in \{0, m\}$. Then,

(i) the *cohesive strategy* C_m is *Nash Equilibrium* if and only if $x \geq C_Y$, and

(ii) the *cohesive strategy* C_0 is *Nash Equilibrium* if and only if $x \leq C_N$, It might be referred to the Nash equilibria interval $I(C_m)$ by $I(Y)$ and the Nash equilibria interval $I(C_0)$ by $I(N)$. The representations of the Nash equilibria intervals $I(Y)$ and $I(N)$ along the horizontal axis determine the decision intervals. The intersections of the intervals $I(Y)$ and $I(N)$ are characterized by the way the cohesive thresholds $C(Y)$ and $C(N)$ are ordered along the real line.

We now state a result concerning the non-existence of cohesive Nash equilibria strategies.

Lemma 2.3. *[9] Assume the decision parameter is such that* $A<0$. *Then there exist a cohesive Nash equilibria strategies for every* $x \notin (C(N), C(Y))$.

2.4 Bifurcated thresholds

In this section, we study the order of the cohesive strategies along the horizontal preference x-axis. Since the cohesive thresholds C_Y and C_N are taking real values, a bifurcation may occur. Such occurrence can be characterized by us in the follows.

Lemma 2.4. *[9] Assume the decision parameter is such that* $A>0$. *Then there exist a cohesive Nash equilibria strategies for every* $x \in (C(Y), C(N))$.

We remark that, if the two cohesive thresholds $C(Y)$ and $C(N)$ are coinciding, then the two cohesive thresholds are in bifurcation position.

Lemma 2.5. *[9] Assume the decision parameter* $A = 0$. *Then there exist a unique cohesive Nash equilibrium for every* $x \in \mathbb{R} \backslash \{x = C(Y) = C(N)\}$.

2.5 Split Nash Equilibrium strategies

In this section, we study the no-cohesive strategies or the split strategies.

Definition 2.5. *A split strategy is a strategy in which players are in favor to make different decisions.*

The *split* strategy is a strategy where the players split into two groups to make two decisions. Thus, there are $(m-1)$ possible split strategies. l *strategy* indicates to l^Y players make decide Y, and so $m - l^Y = l^N$ indicates players make decide N. Note that, a necessary condition for the players to split between the two decisions is that $l^Y \in \{1, 2, \dots, m-1\}$. Our goal determine and characterize all split strategies that form Nash equilibria, by determining the necessary and sufficient conditions that guarantee the existence of split Nash equilibria strategies. We represent the following definition.

Definition 2.6. *The left split threshold $S_L(l^Y)$ for the strategies Y is defined by*

$$S_L(l^Y) = -\alpha^Y(m-1) + (\alpha^Y + \alpha^N)(m - l^Y) \tag{2.6}$$

and the right split threshold $S_R(l^N)$ for the strategies N is defined by

$$S_R(l^N) = \alpha^N(m-1) - (\alpha^Y + \alpha^N)(m - l^N)\ . \tag{2.7}$$

Now we connect the above two thresholds by the following result which explains the cost of moving one player from one decision to another.

Prposition 2.1. *[9] Given $S \in \boldsymbol{S}$ for all $l^Y = 1, 2, \dots, m-1$ we have $S_R(m-(l^Y-1)) = S_L(l^Y)$. Furthermore, $S_R(l^N) - S_L(l^Y) = -(\alpha^Y + \alpha^N)$.*

See Figure 2.1 as an illustration of Lemma 2.1. Based on Figure 2.1, one can see that if $l^Y = 1$, then

$$S_L(1) = C(N)$$

Moreover, we observe that

$$\begin{aligned} S_R(m - l^Y + 1) &= S_R(m) \\ &= \alpha^N(m-1) \\ &= C(N) \end{aligned}$$

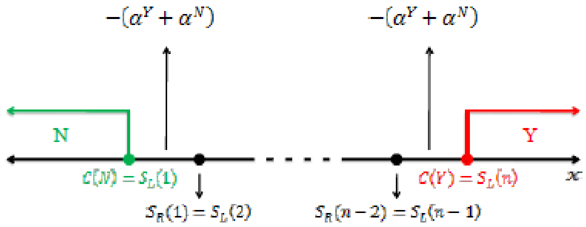


Fig. 2.1: Split Nash domain when $A < 0$.

The following result determines the necessary conditions for a split strategy to be Nash Equilibrium.

Lemma 2.6. *[9] Let $S \in \boldsymbol{S}$ be a split Nash Equilibrium strategy. If $A > 0$ then the players can not be split between the strategies C_m and C_0.*

Now, we introduce the following characterizing result that guarantees the split strategy for players to be *Nash Equilibrium.*

Theorem 2.7. *[9] Assume $S \in \boldsymbol{S}$ is a split strategy. Then, S is Nash equilibrium strategy if and only if the difference decision parameter x is such that*

$$x \in [S_L(l^Y), S_R(l^N)] \ .$$

Definition 2.7. *The matching split thresholds is defined by*

$$G(x) = (-x + \alpha^N(m-1))/A \tag{2.8}$$

and the relative preference split thresholds is defined by

$$X(l) = \alpha^N(m-1) - lA \ . \tag{2.9}$$

Let us define also $Z(x)$ by

$$Z(x) = x - \alpha^N m - \alpha^Y \ . \tag{2.10}$$

We observe that, the map $G(x)$ is a increasing affine functions in x, with the property that $G(C(N)) = 0$ and $G(C(Y)) = m - 1$. Note that

$$\begin{aligned} G(C(N)) &= 0 \\ G(C(N) - A) &= 1 \\ G(C(N) - 2A) &= 2 \\ G(C(N) - 3A) &= 3 \\ &\vdots \\ G(C(N) - (m-1)A) &= G(C(Y)) \\ &= (m-1). \end{aligned}$$

The equality is done by using (2.5). In the following result, we show how the split strategy l is *Nash Equilibrium* if and only if $l \in [G(x), G(x)+1]$ for a given preference $x \in [C(N), C(Y)]$.

Note that, when x is restricted to $C(N)$ or $C(Y)$, l becomes cohesive.

Theorem 2.8. *[9] Assume that the decision parameter $A < 0$. Given the split strategy $S \in \boldsymbol{S}$. The split strategy l is a Nash Equilibrium if and only if $x \in [X(l) + A, X(l)]$.*

2.6 Decision game model with interactions between two types of players

The decision model which formulated in [4] is going to be reviewed in this section. Let $\mathbf{T} = \{t_1, t_2\}$ be set with two types of players, type one has n_1 players in the set $I_1 = \{1, \ldots, n_1\}$ and type two has n_2 players in the set $I_2 = \{1, \ldots, n_2\}$. The disjoint union of I_1 and I_2 is given by $I_1 \sqcup I_2$. Each player $i \in \mathbf{I}$ is assumed to choose one possible decision d in the set $\mathbf{D} = \{Y, N\}$.

Suppose $\omega_p^d \in \mathbb{R}$ describes the taste of player with type $t_p \in \mathbf{T}$

choosing decision $d \in \mathbf{D}$, so the **taste matrix** L is defined as

$$L = \begin{pmatrix} \omega_1^Y & \omega_1^N \\ \omega_2^Y & \omega_2^N \end{pmatrix}.$$

We define N_d as the **crowding matrix** in which the entry $\alpha_{pq}^d \in \mathbb{R}$ describes the crowding effect on player with type t_p by a player with type t_q in choosing $d \in \mathbf{D}$:

$$N_p = \begin{pmatrix} \alpha_{11}^d & \alpha_{12}^d \\ \alpha_{21}^d & \alpha_{22}^d \end{pmatrix}.$$

Each type of players who like or dislike being with in each decision can be pointed out by the preference neighbors matrix, i.e. the players crowding type.

The players choose their favorite (pure) decision according to the strategy

$$S : \mathbf{I} \longrightarrow \mathbf{D} \ .$$

The strategy S indicates for each player $i \in \mathbf{I}$, his favorite choice $S(i) \in \mathbf{D}$. We assume $\mathbf{S}$ to be the set of all possible strategy S.

Under any strategy S, $l_p^d = l_p^d(S)$ refers to the number of players whose type is t_p choosing d. We define O_s as the **strategic decision matrix**

$$O_s = \begin{pmatrix} l_1^Y & l_1^N \\ l_2^Y & l_2^N \end{pmatrix}.$$

If $l_1 = l_1^Y(S)$ is the number of players whose type is t_1 choosing Y under the strategy S, then $l_2 = l_2^Y(S)$ is the number of players whose type is t_2 choosing Y under the strategy S. Therefore, $n_1 - l_1$ is the number of players whose type is t_1 choosing N under the strategy S and $n_2 - l_2$ is the number of players whose type is t_2 choosing N under the strategy S. Let $\mathbf{O}$ be defined by

$$\mathbf{O} = \{0, \ldots, n_1\} \times \{0, \ldots, n_2\}.$$

Let $U_1 : \mathbf{D} \times \mathbf{O} \longrightarrow \mathbb{R}$ be the utility function of player with type t_1 who makes decision Y(resp. N) given as

$$\begin{aligned} U_1(Y; l_1, l_2) &= \omega_1^Y + \alpha_{11}^Y (l_1 - 1) + \alpha_{12}^Y l_2, && (2.11) \\ U_1(N; l_1, l_2) &= \omega_1^N + \alpha_{11}^N (n_1 - l_1 - 1) + \alpha_{12}^N (n_2 - l_2). && (2.12) \end{aligned}$$

Let $U_2 : \mathbf{D} \times \mathbf{O} \longrightarrow \mathbb{R}$ be the utility function of player with type t_2 who makes decision Y(resp. N) given as

$$\begin{aligned} U_2(Y; l_1, l_2) &= \omega_2^Y + \alpha_{22}^Y (l_2 - 1) + \alpha_{21}^Y l_1, && (2.13) \\ U_2(N; l_1, l_2) &= \omega_2^N + \alpha_{22}^N (n_2 - l_2 - 1) + \alpha_{21}^N (n_1 - l_1). && (2.14) \end{aligned}$$

Given a strategy $S \in \mathbf{S}$, the utility $U_i(S)$ of player $i \in \mathbf{I}$ with type t_j is given by

$$U_j(S(i); l_1^Y(S), l_2^Y(S)), \qquad j \in \{1, 2\}.$$

Definition 2.8. *The horizontal taste of players with type t_1 is defined by $x = \omega_1^Y - \omega_1^N$, and the vertical taste of players with type t_2 is defined by $y = \omega_2^Y - \omega_2^N$.*

The following explanation are considered under no influence by other players:

- If $y > 0$, then players whose type is t_2 are in favor to choose Y,
- If $y = 0$, then players whose type is t_2 are indifferent to choose any decision,
- If $y < 0$, then players whose type is t_2 are in favor to choose N. Similarly, the explanation follows for the cases $x > 0$, $x = 0$, and $x < 0$ for players whose type is t_1.

Definition 2.9. *A strategy $S^* : \boldsymbol{I} \longrightarrow \boldsymbol{D}$ is a (pure) Nash Equilibrium iff*

$$U_i(S^*) \geq U_i(S), \forall i \in I$$

and for every strategy $S \in \boldsymbol{S}$.

If $S \in \mathbf{S}$ is NE strategy, then $\mathbf{N}(S)$ is the Nash domain contains all taste pairs (x, y) associated to this strategy.

2.7 Cohesive Nash equilibria

In this section, we show how horizontal taste x and vertical tastes y together with n_1 and n_2 determine the position of the NE.

A strategy where all players are in favor of choosing the same decision is called cohesive. *Non-cohesive strategy* is called desperate.

Under any strategy $S \in \mathbf{S}$, we may observe the following cohesive cases:

- (Y, Y) profile where all players are in favor of choosing Y,
- (Y, N) profile where players whose type t_1 are in favor of choosing Y while players whose type t_2 are in favor of choosing N,
- (N, Y) profile where players whose type t_1 are in favor of choosing N while players whose type t_2 are in favor of choosing Y,
- (N, N) profile where all players are in favor of choosing N.

The Nash domain $\mathbf{N}(Y, Y)$ is given by

$$\mathbf{N}(Y,Y) = \{(x,y) : x \geq H(Y,Y) \quad \text{and} \quad y \geq V(Y,Y)\}, \tag{2.15}$$

where the horizontal and vertical cut points are formed, respectively, by

$$\begin{aligned} H(Y,Y) &= -\alpha_{11}^{Y}(n_1-1) - \alpha_{12}^{Y}n_2, \\ V(Y,Y) &= -\alpha_{22}^{Y}(n_2-1) - \alpha_{21}^{Y}n_1. \end{aligned} \tag{2.16}$$

So, a *cohesive strategy* (Y, Y) is *NE* iff $(x, y) \in \mathbf{N}(Y, Y)$.

To see that, the *cohesive strategy* (Y, Y) is *NE* iff

$$U_1(Y; n_1, n_2) \geq U_1(N; n_1-1, n_2) \quad \text{and} \tag{2.17}$$

$$U_2(Y; n_1, n_2) \geq U_2(N; n_1, n_2-1). \tag{2.18}$$

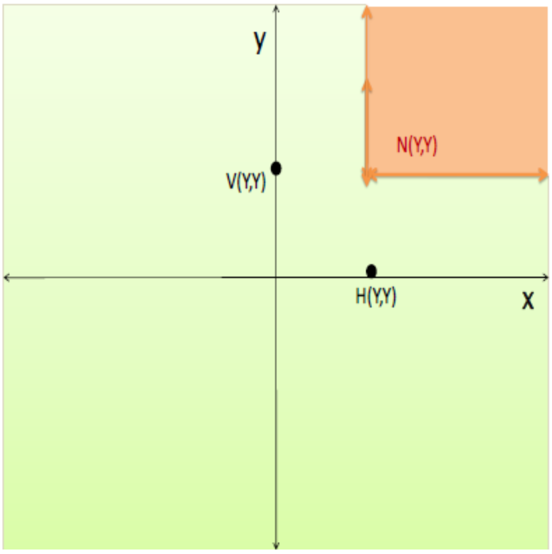


Fig. 2.2: Cohesive Nash equilibria quadrant $\mathbf{N}(Y, Y)$.

Substituting the utility functions given in (2.11) and (2.12) in inequality (2.17) we get

$$\omega_1^Y + \alpha_{11}^Y(n_1 - 1) + \alpha_{12}^Y n_2 \geq \omega_1^N + \alpha_{11}^N(n_1 - 1 - (n_1 - 1)) + \alpha_{12}^N(n_2 - n_2).$$

Rearrange previous inequality, we get

$$\omega_1^Y - \omega_1^N \geq -\alpha_{11}^Y(n_1 - 1) - \alpha_{12}^Y n_2.$$

By simplifying the previous equations, we get $x \geq H(Y, Y)$. Similarly, substituting the utility functions (2.14) and (2.13) in the inequality (2.18), we obtain

$$\omega_2^Y + \alpha_{22}^Y(n_2 - 1) + \alpha_{21}^Y n_1 \geq \omega_2^N + \alpha_{22}^N(n_2 - 1 - (n_2 - 1)) + \alpha_{21}^N(n_1 - n_1).$$

Rearrange the previous inequality, we get

$$\omega_2^Y - \omega_2^N \geq -\alpha_{22}^Y(n_2 - 1) - \alpha_{21}^Y n_1,$$

By simplifying the previous equations, we get $y \geq V(Y, Y)$.

The *Nash domain* $\mathbf{N}(Y, N)$ is given by

$$\mathbf{N}(Y, N) = \{(x, y) : x \geq H(Y, N) \quad \text{and} \quad y \leq V(Y, N)\}, \tag{2.19}$$

where the horizontal and vertical cut points are formed, respectively, by

$$H(Y,N) = -\alpha_{11}^{Y}(n_1-1)+\alpha_{12}^{N}n_2, \quad (2.20)$$
$$V(Y,N) = \alpha_{22}^{N}(n_2-1)-\alpha_{21}^{Y}n_1. \quad (2.21)$$

So, a *cohesive strategy* (Y,N) is *NE* iff $(x,y) \in \mathbf{N}(Y,N)$. To see that,

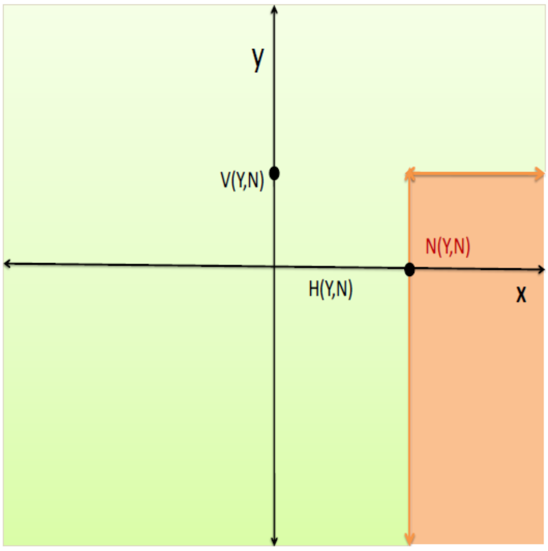


Fig. 2.3: Cohesive Nash equilibria quadrant $\mathbf{N}(Y,N)$.

the *cohesive strategy* (Y,N) is *NE* iff

$$U_1(Y;n_1,0) \geq U_1(N;n_1-1,0) \quad \text{and} \quad (2.22)$$
$$U_2(N;n_1,0) \geq U_2(Y;n_1,1). \quad (2.23)$$

Substituting the utility functions given in (2.11) and (2.12) in the inequality (2.22), we obtain

$$\omega_1^Y+\alpha_{11}^{Y}(n_1-1)+\alpha_{12}^{Y}(0) \geq \omega_1^N+\alpha_{11}^{N}(n_1-(n_1-1)-1)+\alpha_{12}^{N}(n_2-0).$$

Rearrange the previous inequality, we get

$$\omega_1^Y-\omega_1^N \geq -\alpha_{11}^{Y}(n_1-1)+\alpha_{12}^{N}n_2,$$

which simplifies to $x \geq H(Y,N)$. Similarly, substituting the utility functions (2.14) and (2.13) in the inequality (2.23), we obtain

$$\omega_2^N+\alpha_{22}^{N}(n_2-0-1)+\alpha_{21}^{N}(n_1-n_1) \geq \omega_2^Y+\alpha_{22}^{Y}(1-1)+\alpha_{21}^{Y}n_1.$$

Rearrange the previous inequality, we get

$$\omega_2^Y - \omega_2^N \leq \alpha_{22}^N(n_2 - 1) - \alpha_{21}^N(n_1),$$

which simplifies to $y \leq V(Y, N)$.

The *Nash domain* $\mathbf{N}(N, Y)$ is given by

$$\mathbf{N}(N, Y) = \{(x, y) : x \leq H(N, Y) \quad \text{and} \quad y \geq V(N, Y)\}, \tag{2.24}$$

where the horizontal and vertical cut points are formed, respectively, by

$$\begin{aligned} H(N, Y) &= \alpha_{11}^N(n_1 - 1) - \alpha_{12}^Y n_2, \\ V(N, Y) &= -\alpha_{22}^Y(n_2 - 1) + \alpha_{21}^N n_1. \end{aligned} \tag{2.25}$$

So, a *cohesive strategy* (N, Y) is *NE* iff $(x, y) \in \mathbf{N}(N, Y)$. To see that,

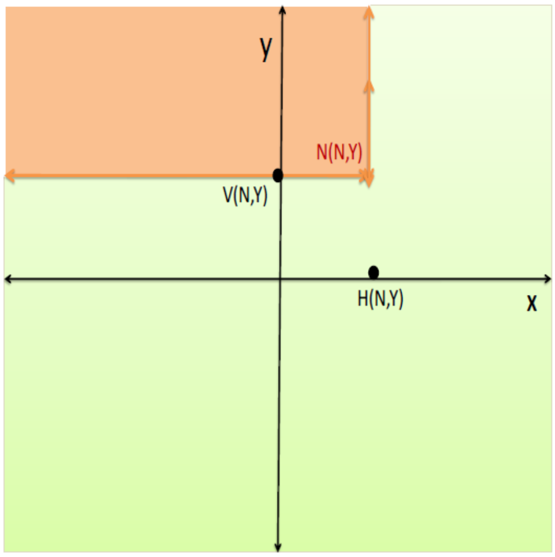


Fig. 2.4: Cohesive Nash equilibria quadrant $\mathbf{N}(N, Y)$.

the *cohesive strategy* (N, Y) is *NE* iff

$$U_1(N; 0, n_2) \geq U_1(Y; 1, n_2) \quad \text{and} \tag{2.26}$$

$$U_2(Y; 0, n_2) \geq U_2(N; 0, n_2 - 1). \tag{2.27}$$

Substituting the utility functions given in (2.11) and (2.12) in the inequality (2.26), we obtain

$$\omega_1^N + \alpha_{11}^N(n_1 - 0 - 1) + \alpha_{12}^N(n_2 - n_2) \geq \omega_1^Y + \alpha_{11}^Y(1 - 1) + \alpha_{12}^Y n_2.$$

Rearrange the previous inequality, we get

$$\omega_1^Y - \omega_1^N \leq \alpha_{11}^N(n_1 - 1) - \alpha_{12}^Y n_2,$$

which simplifies to $x \leq H(N, Y)$. similarly, by substituting the utility functions (2.14) and (2.13) in the inequality (2.27), we obtain

$$\omega_2^Y + \alpha_{22}^Y(n_2 - 1) + \alpha_{21}^Y(0) \geq \omega_2^N + \alpha_{22}^N(n_2 - (n_2 - 1) - 1) + \alpha_{21}^N n_1.$$

Rearrange the previous inequality, we get

$$\omega_2^Y - \omega_2^N \geq -\alpha_{22}^N(n_2 - 1) + \alpha_{21}^Y n_1,$$

which simplifies to $y \geq V(Y, N)$.

The *Nash domain* $\mathbf{N}(N, N)$ is given by

$$\mathbf{N}(N, N) = \{(x, y) : x \leq H(N, N) \quad \text{and} \quad y \leq V(N, N)\}, \quad (2.28)$$

where the horizontal and vertical cut points are formed, respectively, by

$$\begin{aligned} H(N, N) &= \alpha_{11}^N(n_1 - 1) + \alpha_{12}^N n_2, \\ V(N, N) &= \alpha_{22}^N(n_2 - 1) + \alpha_{21}^N n_1. \end{aligned} \quad (2.29)$$

So, a *cohesive strategy* (N, N) is *NE* iff $(x, y) \in \mathbf{N}(N, N)$. To see that, the *cohesive strategy* (N, N) is *NE* iff

$$U_1(N; 0, 0) \geq U_1(Y; 1, 0) \quad \text{and} \quad (2.30)$$

$$U_2(N; 0, 0) \geq U_2(Y; 0, 1). \quad (2.31)$$

Substituting the utility functions given in (2.11) and(2.12) in the inequality (2.30), we obtain

$$\omega_1^N + \alpha_{11}^N(n_1 - (0) - 1) + \alpha_{12}^N(n_2 - 0) \leq \omega_1^Y + \alpha_{11}^Y(1 - 1) + \alpha_{12}^Y(0).$$

Rearrange the previous inequality, we get

$$\omega_1^Y - \omega_1^N \leq \alpha_{11}^N(n_1 - 1) + \alpha_{12}^N n_2,$$

which simplifies to $x \leq H(N, N)$. Similarly, substituting the utility functions (2.14) and (2.13) in the inequality (2.31), we obtain

$$\omega_2^N + \alpha_{22}^N(n_2 - 0 - 1) + \alpha_{21}^N(n_1) \geq \omega_2^Y + \alpha_{22}^Y(1 - 1) + \alpha_{21}^Y(0).$$

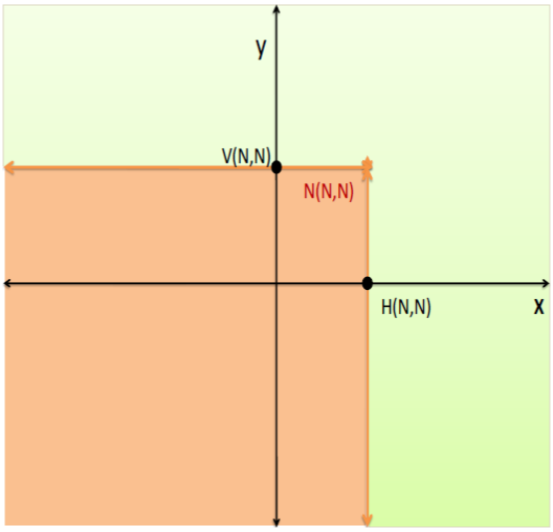


Fig. 2.5: Cohesive Nash equilibria quadrant $\mathbf{N}(N, N)$.

Rearrange the previous inequality, we get

$$\omega_2^Y - \omega_2^N \leq \alpha_{22}^N(n_2 - 1) + \alpha_{21}^Y n_1,$$

which simplifies to $y \leq V(N, N)$.

2.8 Split Nash equilibria

Under any strategy $S \in \mathbf{S}$. Recall $l_1(S) = l_1$ and $l_2(S) = l_2$. The strategy profile (l_1, l_2) is cohesive iff $l_1 \in \{0, n_1\}$ and $l_2 \in \{0, n_2\}$. If $l_1 \in \{1, 2, \ldots, n_1 - 1\}$ or $l_2 \in \{1, 2, \ldots, n_2 - 1\}$, then (l_1, l_2) is split strategy profile. More case of split occur when $l_1 \in \{0, n_1\}$ but $l_2 \notin \{0, n_2\}$ or when $l_2 \in \{0, n_2\}$ but $l_1 \notin \{0, n_1\}$.

Definition 2.10. *Let A be the influence crowding matrix:*

$$\begin{aligned} A &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \\ &= \begin{pmatrix} \alpha_{11}^Y + \alpha_{11}^N & \alpha_{12}^Y + \alpha_{12}^N \\ \alpha_{21}^Y + \alpha_{21}^N & \alpha_{22}^Y + \alpha_{22}^N \end{pmatrix}. \end{aligned}$$

Given a strategy decision $S \in \mathbf{S}$. The (pure) Nash region $N(l_1, l_2)$ consists of all tastes (x, y) that guarantee the strategy profile (l_1, l_2) to be *NE*. We now proceed to the following result.

Lemma 2.9. *[9] Assume (l_1, l_2) is NE.*

(i) If $A_{11} > 0$, then $l_1 \in \{0, n_1\}$.

(ii) If $A_{22} > 0$, then $l_2 \in \{0, n_2\}$.

Furthermore, if A_{11} and A_{22} are both positive, then (l_1, l_2) is cohesive.

Chapter 3

Characterizing the geometry of envy human behavior using game theoretical model with two types of homogeneous players

In this chapter, we model the influence of envy behavior created by both types of players over the utility function of each other and study how this influence changes the Cartesian position of the Nash equilibria studied in [8].

Let $\beta_i > 0$, $i = 1, 2$ be the envy parameter associated with players of type t_i. We remark that, β_1 (resp. β_2) measures the influence of the envy behavior created by players with type t_1 (resp. t_2) over the utility function of players with type t_2 (resp. t_1). Furthermore, we assume that β_1 and β_2 do not depend on the decision d which has been made by players with type t_1 and t_2, respectively. However, a general framework includes such dependence could be studied in a different model.

Let $U_1^{\mathbf{e}} : \mathbf{D} \times \mathbf{O} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ be the utility function of an envy player with type t_1 who makes decision Y given by

$$\begin{aligned} U_1^{\mathbf{e}}(Y; l_1, l_2, \beta_1) &< U_1(Y; l_1, l_2) - \beta_1 U_2(Y; l_1, l_2) \\ &= \omega_1^Y + \alpha_{11}^Y (l_1 - 1) + \alpha_{12}^Y l_2 \\ &\quad -\beta_1(\omega_2^Y + \alpha_{22}^Y (l_2 - 1) + \alpha_{21}^Y l_1) \end{aligned} \tag{3.1}$$

and let $U_1^{\mathbf{e}} : \mathbf{D} \times \mathbf{O} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ be the utility function of an envy player with type t_1 who makes decision N given by

$$\begin{aligned} U_1^{\mathbf{e}}(N; l_1, l_2, \beta_1) &= U_1(N; l_1, l_2) - \beta_1 U_2(N; l_1, l_2) \\ &= \omega_1^N + \alpha_{11}^N (n_1 - l_1 - 1) + \alpha_{12}^N (n_2 - l_2) \\ &\quad -\beta_1(\omega_2^N + \alpha_{22}^N (n_2 - l_2 - 1) + \alpha_{21}^N (n_1 - l_1)) \end{aligned} \tag{3.2}$$

Let $U_2^{\mathbf{e}} : \mathbf{D} \times \mathbf{O} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ be the utility function of an envy player with type t_2 who makes decision Y given by

$$\begin{aligned} U_2^{\mathbf{e}}(Y; l_1, l_2, \beta_2) &= U_2(Y; l_1, l_2) - \beta_2 U_1(Y; l_1, l_2) \\ &= \omega_2^Y + \alpha_{22}^Y (l_2 - 1) + \alpha_{21}^Y l_1 \\ &\quad -\beta_2(\omega_1^Y + \alpha_{11}^Y (l_1 - 1) + \alpha_{12}^Y l_2) \end{aligned} \tag{3.3}$$

and let $U_2^{\mathbf{e}} : \mathbf{D} \times \mathbf{O} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ be the utility function of an envy player with type t_2 who makes decision N given by

$$\begin{aligned} U_2^{\mathbf{e}}(N; l_1, l_2, \beta_2) &= U_2(N; l_1, l_2) - \beta_2 U_1(N; l_1, l_2) \\ &= \omega_2^N + \alpha_{22}^N (n_2 - l_2 - 1) + \alpha_{21}^N (n_1 - l_1) \\ &\quad -\beta_2(\omega_1^N + \alpha_{11}^N (n_1 - l_1 - 1) + \alpha_{12}^N (n_2 - l_2)), \end{aligned} \tag{3.4}$$

where the utility functions $U_i(d; l_1, l_2)$, $i = 1, 2$ and $d \in D$ are as given in (2.11),(2.12),(2.14) and (2.13). We remark that, if $\beta_1 = \beta_2 = 0$, then the envy model coincides with the decision model presented in [10].

3.1 *Geometry of pure envy Nash equilibria*

In this section, we study how the horizontal and vertical tastes x and y, the coefficients of the crowding matrix A together with n_1 and n_2 determine the cohesive ENE (Envy Nash Equilibria).

Definition 3.1. *A strategy* $S^*_{\mathbf{e}} : \boldsymbol{I} \longrightarrow \boldsymbol{D}$ *is a (pure) ENE iff*

$$U_i(S^*_{\mathbf{e}}) \geq U_i(S) \quad \forall i \in \boldsymbol{I}$$

and for every strategy $S \in \boldsymbol{S}$.

The *envy Nash domain* $\mathbf{N}^{\mathbf{e}}(S)$ of a strategy $S \in \mathbf{S}$ contains all taste pairs (x, y) that guarantee the strategy S is an *envy Nash Equilibrium.*

Definition 3.2. *A (pure) envy strategy where all players are in favor of choosing same decision is called envy cohesive strategy. A (pure) envy strategy that is not cohesive is called envy desperate strategy.*

We observe that there are four distinct envy cohesive strategies. We now construct the envy Nash domains $\mathbf{N}^{\mathbf{e}}(S_{\mathbf{e}})$ for each pure envy strategy $S_{\mathbf{e}} \in \mathbf{S}$. The four envy Nash domains are $\mathbf{N}^{\mathbf{e}}(Y, Y)$, $\mathbf{N}^{\mathbf{e}}(Y, N)$, $\mathbf{N}^{\mathbf{e}}(N, Y)$ and $\mathbf{N}^{\mathbf{e}}(N, N)$.

Theorem 3.1. *Assume that* $\beta_1\beta_2<1$.

(i) The envy cohesive strategy $S_{\mathbf{e}} = (Y, Y)$ *is ENE iff* $(x, y) \in \boldsymbol{N}^{\mathbf{e}}(Y, Y)$, *the envy Nash region* $\boldsymbol{N}^{\mathbf{e}}(Y, Y)$ *is*

$$\boldsymbol{N}^{\mathbf{e}}(Y, Y) = \{(x, y) \in \mathbb{R}^2 : x \geq H^{\mathbf{e}}(Y, Y) \quad and \quad y \geq V^{\mathbf{e}}(Y, Y)\}, \tag{3.5}$$

and the horizontal envy and vertical envy cut points are formed, respectively, by

$$\begin{aligned} H^{\mathbf{e}}(Y, Y) &= H(Y, Y) + \beta_1 \left[\frac{\alpha^N_{22} - \alpha^N_{21} + \beta_2(\alpha^N_{11} - \alpha^N_{12})}{1 - \beta_1\beta_2}\right] \\ V^{\mathbf{e}}(Y, Y) &= V(Y, Y) + \beta_2 \left[\frac{\beta_1(\alpha^N_{22} - \alpha^N_{21}) + \alpha^N_{11} - \alpha^N_{12}}{1 - \beta_1\beta_2}\right], \end{aligned} \tag{3.6}$$

where the cut points $H(Y, Y)$ *and* $V(Y, Y)$ *are as given in the second inequality of* (2.16).

(ii) The envy cohesive strategy $S_{\mathbf{e}} = (Y, N)$ *is ENE iff* $(x, y) \in N^{\mathbf{e}}(Y, N)$, *the envy Nash region* $\boldsymbol{N}^{\mathbf{e}}(Y, N)$ *is*

$$\boldsymbol{N}^{\mathbf{e}}(Y, N) = \{(x, y) : x \geq H^{\mathbf{e}}(Y, N) \quad and \quad y \leq V^{\mathbf{e}}(Y, N)\} \tag{3.7}$$

and the horizontal envy and vertical envy cut points are formed, respectively, by

$$
\begin{aligned}
H^{\mathbf{e}}(Y,N) &= H(Y,N) + \beta_1 \left[\frac{\beta_2(\alpha_{11}^N + \alpha_{12}^Y) - (\alpha_{22}^Y + \alpha_{21}^N)}{1-\beta_1\beta_2}\right] \\
V^{\mathbf{e}}(Y,N) &= V(Y,N) + \beta_2 \left[\frac{\alpha_{11}^N + \alpha_{12}^Y - \beta_1(\alpha_{22}^Y + \alpha_{21}^N)}{1-\beta_1\beta_2}\right],
\end{aligned} \tag{3.8}
$$

where the cut points $H(Y,N)$ and $V(Y,N)$ are as given in the inequality (2.21).

(iii) *The envy cohesive strategy $S_{\mathbf{e}} = (N,Y)$ is ENE iff $(x,y) \in \boldsymbol{N}^{\mathbf{e}}(N,Y)$, the envy Nash region $\boldsymbol{N}^{\mathbf{e}}(N,Y)$ is*

$$\boldsymbol{N}^{\mathbf{e}}(N,Y) = \{(x,y) : x \leq H^{\mathbf{e}}(N,Y) \quad \text{and} \quad y \geq V^{\mathbf{e}}(N,Y)\} \tag{3.9}$$

and the horizontal envy and vertical envy cut points are formed, respectively, by

$$
\begin{aligned}
H^{\mathbf{e}}(N,Y) &= H(N,Y) + \beta_1 \left[\frac{\alpha_{22}^N + \alpha_{21}^Y - \beta_2(\alpha_{11}^Y + \alpha_{12}^N)}{1-\beta_1\beta_2}\right] \\
V^{\mathbf{e}}(N,Y) &= V(N,Y) + \beta_2 \left[\frac{\beta_1(\alpha_{22}^N + \alpha_{21}^Y) - (\alpha_{11}^Y + \alpha_{12}^N)}{1-\beta_1\beta_2}\right],
\end{aligned} \tag{3.10}
$$

where the cut points $H(N,Y)$ and $V(N,Y)$ are as given in the second inequality(2.25).

(iv) *The envy cohesive strategy $S_{\mathbf{e}} = (N,N)$ is ENE iff $(x,y) \in \boldsymbol{N}^{\mathbf{e}}(N,N)$, the envy Nash region $\boldsymbol{N}^{\mathbf{e}}(N,N)$ is*

$$\boldsymbol{N}^{\mathbf{e}}(N,N) = \{(x,y) : x \leq H^{\mathbf{e}}(N,N) \quad \text{and} \quad y \leq V^{\mathbf{e}}(N,N)\}, \tag{3.11}$$

and the horizontal envy and vertical envy cut points are formed, respectively, by

$$
\begin{aligned}
H^{\mathbf{e}}(N,N) &= H(N,N) + \beta_1 \left[\frac{\alpha_{21}^Y - \alpha_{22}^Y + \beta_2(\alpha_{12}^Y - \alpha_{11}^Y)}{1-\beta_1\beta_2}\right] \\
V^{\mathbf{e}}(N,N) &= V(N,N) + \beta_2 \left[\frac{\beta_1(\alpha_{21}^Y - \alpha_{22}^Y) + \alpha_{12}^Y - \alpha_{11}^Y}{1-\beta_1\beta_2}\right],
\end{aligned} \tag{3.12}
$$

where the cut points $H(N,N)$ and $V(N,N)$ are as given in the second inequality of (2.29)*.*

Proof. It is enough to prove cases (i)-(ii). The cohesive envy strategy $S_{\mathbf{e}} = (Y, Y)$ is ENE iff

$$\begin{aligned} U_1(Y^{\mathbf{e}}; n_1, n_2, \beta_1) &\geq U_1(N^{\mathbf{e}}; n_1 - 1, n_2, \beta_1) \qquad \text{and} \\ U_2(Y^{\mathbf{e}}; n_1, n_2, \beta_2) &\geq U_2(N^{\mathbf{e}}; n_1, n_2 - 1, \beta_2). \end{aligned} \tag{3.13}$$

Substituting the envy utility functions in the first inequality of (3.13), we obtain

$$\begin{aligned} &\omega_1^Y + \alpha_{11}^Y(n_1 - 1) + \alpha_{12}^Y n_2 - \beta_1(\omega_2^Y + \alpha_{22}^Y(n_2 - 1) + \alpha_{21}^Y n_1) \geq \\ &\omega_1^N + \alpha_{11}^N(n_1 - 1 - (n_1 - 1)) + \alpha_{12}^N(n_2 - n_2) \\ &-\beta_1[\omega_2^N + \alpha_{22}^N(n_2 - n_2 - 1) + \alpha_{21}^N(n_1 - (n_1 - 1))] \end{aligned}$$

Rearrange the above inequality, we get

$$\begin{aligned} \omega_1^Y - \omega_1^N - \beta_1\omega_2^Y + \beta_1\omega_2^N \geq &- \alpha_{11}^Y(n_1 - 1) - \alpha_{12}^Y n_2 \\ &+ \beta_1[\alpha_{22}^Y(n_2 - 1) + \alpha_{21}^Y n_1 + \alpha_{22}^N - \alpha_{21}^N]. \end{aligned}$$

By simplifying the previous equations, we get

$$\begin{aligned} x - \beta_1 y \geq &- \alpha_{11}^Y(n_1 - 1) - \alpha_{12}^Y n_2 \\ &+ \beta_1[\alpha_{22}^Y(n_2 - 1) + \alpha_{21}^Y n_1 + \alpha_{22}^N - \alpha_{21}^N]. \end{aligned} \tag{3.14}$$

Similarly, substituting the utility functions in the second inequality of (3.13), we obtain

$$\begin{aligned} &\omega_2^Y + \alpha_{22}^Y(n_2 - 1) + \alpha_{21}^Y n_1 - \beta_2(\omega_1^Y + \alpha_{11}^Y(n_1 - 1) + \alpha_{12}^Y n_2) \geq \\ &\omega_2^N + \alpha_{22}^N(n_2 - 1 - (n_2 - 1)) + \alpha_{21}^N(n_1 - n_1) \\ &-\beta_2[\omega_1^N + \alpha_{11}^N(n_1 - n_1 - 1) + \alpha_{12}^N(n_2 - (n_2 - 1))]. \end{aligned}$$

Rearrange the above inequality, we get

$$\begin{aligned} \omega_2^Y - \omega_2^N - \beta_2\omega_1^Y + \beta_2\omega_1^N \geq &- \alpha_{22}^Y(n_2 - 1) - \alpha_{21}^Y n_1 \\ &+ \beta_2(\alpha_{11}^Y(n_1 - 1) + \alpha_{12}^Y n_2 + \alpha_{11}^N + \alpha_{12}^N). \end{aligned}$$

By simplifying the previous equations, we get

$$\begin{aligned} y - \beta_2 x \geq \ & -\alpha_{22}^{Y}(n_2 - 1) - \alpha_{21}^{Y} n_1 \\ & +\beta_2(\alpha_{11}^{Y}(n_1 - 1) + \alpha_{12}^{Y} n_2 + \alpha_{11}^{N} + \alpha_{12}^{N}). \end{aligned} \tag{3.15}$$

We multiply (3.15) by β_1, then subtract (3.15) from (3.14), we get

$$\begin{aligned} x(1 - \beta_1\beta_2) \geq \ & (-\alpha_{11}^{Y}(n_1 - 1) - \alpha_{12}^{Y} n_2)(1 - \beta_1\beta_2) \\ & +\beta_1 \left[\alpha_{22}^{N} - \alpha_{21}^{N} + \beta_2(\alpha_{11}^{N} - \alpha_{12}^{N})\right]. \end{aligned} \tag{3.16}$$

Substituting the envy utility functions from (3.1) and (3.3) in (3.13) and rearrange the terms, we obtain

$$x \geq -\alpha_{11}^{Y}(n_1 - 1) - \alpha_{12}^{Y} n_2 + \beta_1 \left[\frac{\alpha_{22}^{N} - \alpha_{21}^{N} + \beta_2(\alpha_{11}^{N} - \alpha_{12}^{N})}{1 - \beta_1\beta_2}\right],$$

which simplifies to $x \geq H^{\mathbf{e}}(Y, Y)$. Similarly, we multiply (3.14) by β_2, then subtract (3.14) from (3.15), we get

$$\begin{aligned} y(1 - \beta_1\beta_2) \geq \ & (-\alpha_{22}^{Y}(n_2 - 1) - \alpha_{21}^{Y} n_1)(1 - \beta_1\beta_2) \\ & +\beta_2 \left[\beta_1(\alpha_{22}^{N} - \alpha_{21}^{N}) + \alpha_{11}^{N} - \alpha_{12}^{N}\right] \end{aligned}$$

We could use the abbreviation of $V(Y, Y)$ as follows:

$$y \geq -\alpha_{22}^{Y}(n_2 - 1) - \alpha_{21}^{Y} n_1 + \beta_2 \left[\frac{\beta_1(\alpha_{22}^{N} - \alpha_{21}^{N}) + \alpha_{11}^{N} - \alpha_{12}^{N}}{1 - \beta_1\beta_2}\right].$$

This, respectively, simplifies to

$$x \geq H^{\mathbf{e}}(Y, Y) \quad \text{and} \quad y \geq V^{\mathbf{e}}(Y, Y).$$

Hence, the *envy cohesive strategy* $S_{\mathbf{e}} = (Y, Y)$ is an envy Nash Equilibrum if and only if $(x, y) \in \mathbf{N}^{\mathbf{e}}(Y, Y)$.
The cohesive envy strategy $S_{\mathbf{e}} = (Y, N)$ is *Nash Equilibrium* if and only if the following inequalities hold

$$\begin{aligned} U_1(Y; n_1, 0, \beta_1) \ & \geq U_1(N; n_1 - 1, 0, \beta_1) \qquad \text{and} \\ U_2(N; n_1, 0, \beta_2) \ & \geq U_2(Y; n_1, 1, \beta_2). \end{aligned} \tag{3.17}$$

Substituting the envy utility functions (3.1) and (3.2) in the first inequality of (3.17), we obtain

$$\begin{aligned} & \omega_1^{Y} + \alpha_{11}^{Y}(n_1 - 1) + \alpha_{12}^{Y}(0) - \beta_1(\omega_2^{Y} + \alpha_{22}^{Y}(0 - 1) + \alpha_{21}^{Y} n_1) \quad \geq \\ & \omega_1^{N} + \alpha_{11}^{N}(n_1 - (n_1 - 1) - 1) + \alpha_{12}^{N}(n_2 - 0) \\ & -\beta_1[\omega_2^{N} + \alpha_{22}^{N}(n_2 - 0 - 1) + \alpha_{21}^{N}(n_1 - (n_1 - 1))]. \end{aligned}$$

Rearrange the above inequality, we get

$$\begin{aligned}\omega_1^Y - \omega_1^N - \beta_1\omega_2^Y + \beta_1\omega_2^N \geq & - \alpha_{11}^Y(n_1-1) + \alpha_{12}^N n_2 \\ & + \beta_1[-\alpha_{22}^Y + \alpha_{21}^Y n_1 - \alpha_{22}^N(n_2-1) - \alpha_{21}^N].\end{aligned}$$

By simplifying the previous equations, we get

$$\begin{aligned}x - \beta_1 y \geq & - \alpha_{11}^Y(n_1-1) + \alpha_{12}^N n_2 \\ & + \beta_1[-\alpha_{22}^Y + \alpha_{21}^Y n_1 - \alpha_{22}^N(n_2-1) - \alpha_{21}^N].\end{aligned} \tag{3.18}$$

Similarly, substituting the envy utility functions (3.3) and(3.4) in the second inequality of (3.17), we obtain

$$\begin{aligned}&\omega_2^N + \alpha_{22}^N(n_2-0-1) + \alpha_{21}^N(n_1-n_1) \\ &-\beta_2[\omega_1^N + \alpha_{11}^N(n_1-n_1-1) + \alpha_{12}^N(n_2-0)] \geq \\ &\omega_2^Y + \alpha_{22}^Y(1-1) + \alpha_{21}^Y n_1 - \beta_2[\omega_1^Y + \alpha_{11}^Y(n_1-1) + \alpha_{12}^Y].\end{aligned}$$

By simplifying the previous equations, we get

$$\begin{aligned}y - \beta_2 x \leq\ & \alpha_{22}^N(n_2-1) - \alpha_{21}^Y n_1 \\ & +\beta_2[\alpha_{11}^N - \alpha_{12}^N(n_2) + \alpha_{11}^Y(n_1-1) + \alpha_{12}^Y].\end{aligned} \tag{3.19}$$

We get y from (3.19), and then substituting it in (3.18), we obtain

$$\begin{aligned}x - \beta_1\beta_2 x \geq & - \alpha_{11}^Y(n_1-1) + \alpha_{12}^N n_2 \\ & + \beta_1\beta_2[\alpha_{11}^N - \alpha_{12}^N(n_2) + \alpha_{11}^Y(n_1-1) + \alpha_{12}^Y] \\ & + \beta_1[-\alpha_{22}^Y + \alpha_{21}^Y n_1 - \alpha_{22}^N(n_2-1) - \\ & \qquad \alpha_{21}^N + \alpha_{22}^N(n_2-1) - \alpha_{21}^Y n_1].\end{aligned}$$

Rearrange the above inequality, we get

$$\begin{aligned}x(1-\beta_1\beta_2) \geq\ & (-\alpha_{11}^Y(n_1-1) + \alpha_{12}^N n_2)(1-\beta_1\beta_2) \\ & +\beta_1[-\alpha_{21}^N - \alpha_{22}^Y + \beta_2(\alpha_{11}^N + \alpha_{12}^Y)].\end{aligned} \tag{3.20}$$

We divide inequality (3.20) by $(1-\beta_1\beta_2)$ and noting that $\beta_1\beta_2<1$, we obtain

$$x \geq -\alpha_{11}^Y(n_1-1) + \alpha_{12}^N n_2 + \beta_1\left[\frac{\beta_2(\alpha_{11}^N + \alpha_{12}^Y) - (\alpha_{22}^Y + \alpha_{21}^N)}{1-\beta_1\beta_2}\right].$$

We get x from (3.18), and then substituting it in (3.19), we obtain

$$\begin{aligned} y(1-\beta_1\beta_2) \leq \quad & \alpha_{22}^N(n_2-1) - \alpha_{21}^Y n_1 \\ & +\beta_1\beta_2[-\alpha_{22}^Y + \alpha_{21}^Y n_1 - \alpha_{22}^N(n_2-1) + \alpha_{21}^N] \\ & +\beta_2[\alpha_{11}^N - \alpha_{12}^N n_2 + \alpha_{11}^Y(n_1-1) \\ & \quad +\alpha_{12}^Y + \alpha_{12}^N n_2 - \alpha_{11}^Y(n_1-1)]. \end{aligned}$$

Rearrange the above inequality, we get

$$\begin{aligned} y(1-\beta_1\beta_2) \leq \quad & (\alpha_{22}^N(n_2-1) - \alpha_{21}^Y n_1)(1-\beta_1\beta_2) \\ & +\beta_2[-\beta_1(\alpha_{22}^Y + \alpha_{21}^N) + \alpha_{11}^N + \alpha_{12}^Y]. \end{aligned} \tag{3.21}$$

We divide inequality (3.21) by $(1-\beta_1\beta_2)$ and noting that $\beta_1\beta_2<1$, we obtain

$$y \leq \alpha_{22}^N(n_2-1) - \alpha_{21}^Y n_1 + \beta_2\left[\frac{\alpha_{11}^N + \alpha_{12}^Y - \beta_1(\alpha_{22}^Y + \alpha_{21}^N)}{1-\beta_1\beta_2}\right].$$

This, respectively, simplifies to

$$x \geq H^{\mathbf{e}}(Y,N) \quad \text{and} \quad y \leq V^{\mathbf{e}}(Y,N).$$

Hence, $S_{\mathbf{e}}(Y,N)$ is ENE iff $(x,y) \in \mathbf{N}^{\mathbf{e}}(Y,N)$. □

Now, we study the influence of the envy parameters created by both types of players on the location of Nash equilibria. More precisely, when a certain *Nash Equilibrium* strategy can be *envy Nash Equilibrium* by comparing the Nash domains $\mathbf{N}(S)$ with The envy Nash domains $\mathbf{N}^{\mathbf{e}}(S)$ for a given strategy $S \in \mathbf{S}$.

Lemma 3.2. *Given a strategy $S \in \boldsymbol{S}$. If $S=(Y,Y)$ is a Nash Equilibrium, then it is an envy Nash Equilibrium if and only if*

$$\beta_1(\alpha_{22}^N - \alpha_{21}^N)<\alpha_{12}^N - \alpha_{11}^N \quad \textit{and} \quad \beta_2(\alpha_{11}^N - \alpha_{12}^N)<\alpha_{21}^N - \alpha_{22}^N.$$

Proof. The proof follows from the definitions of the envy Nash domain $\mathbf{N}^{\mathbf{e}}(Y,Y)$ given in (3.5) and the Nash domain $\mathbf{N}(Y,Y)$ given in (2.15) when $\mathbf{N}(Y,Y) \subset \mathbf{N}^{\mathbf{e}}(Y,Y)$.

$$H(Y,Y)>H^e(Y,Y) \qquad \text{and} \qquad V(Y,Y)>V^e(Y,Y).$$

Substituting the thresholds horizontal and vertical strategy

$$\begin{aligned} H(Y,Y) &> H(Y,Y)+\beta_1\left[\frac{\alpha_{22}^N-\alpha_{21}^N+\beta_2(\alpha_{11}^N-\alpha_{12}^N)}{1-\beta_1\beta_2}\right] \\ V(Y,Y) &> V(Y,Y)+\beta_2\left[\frac{\beta_1(\alpha_{22}^N-\alpha_{21}^N)+\alpha_{11}^N-\alpha_{12}^N}{1-\beta_1\beta_2}\right]. \end{aligned}$$

Rearrange the above inequality, we get

$$0>\alpha_{22}^N-\alpha_{21}^N+\beta_2(\alpha_{11}^N-\alpha_{12}^N)$$

and

$$0>\beta_1(\alpha_{22}^N-\alpha_{21}^N)+\alpha_{11}^N-\alpha_{12}^N.$$

Rearranging the terms in the previous inequalities, we get

$$\beta_1(\alpha_{22}^N-\alpha_{21}^N)<\alpha_{12}^N-\alpha_{11}^N \quad \text{and} \quad \beta_2(\alpha_{11}^N-\alpha_{12}^N)<\alpha_{21}^N-\alpha_{22}^N.$$

□

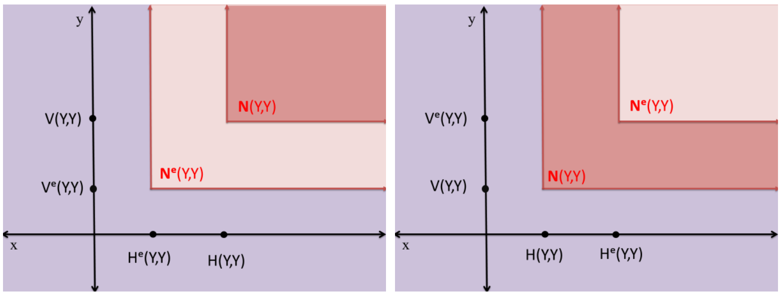
Hence, if players with type t_1(resp. t_2) like more being with players with type t_2(resp. t_1) than being together making decision N (means $\alpha_{11}^N<\alpha_{12}^N$ and $\alpha_{22}^N<\alpha_{21}^N$), then $\mathbf{N}(Y,Y)\subset\mathbf{N}^e(Y,Y)$ holds (see Figure 3.1a) and the following inequalities hold

$$\beta_1>0>\frac{\alpha_{12}^N-\alpha_{11}^N}{\alpha_{22}^N-\alpha_{21}^N} \quad \text{and} \quad \beta_2>0>\frac{\alpha_{22}^N-\alpha_{21}^N}{\alpha_{12}^N-\alpha_{11}^N}.$$

On the other hand, if players with type t_1(resp. t_2) like more being together than being with players with type t_2(resp. t_1) making decision N (means $\alpha_{11}^N>\alpha_{12}^N$ and $\alpha_{22}^N>\alpha_{21}^N$), then $\mathbf{N}^e(Y,Y)\subset\mathbf{N}(Y,Y)$ (see figure 3.1b).

We remark that, Lemma 3.2 provides some properties for the Nash domains $\mathbf{N}^e(Y,Y)$ and $\mathbf{N}(Y,Y)$:

(i) $\mathbf{N}^e(Y,Y)=\mathbf{N}(Y,Y)$ if $\alpha_{11}^N=\alpha_{12}^N$ and $\alpha_{22}^N=\alpha_{21}^N$, which means that the equilibria coincide.
To clarify item (i), we use the horizontal envy $H^{\mathbf{e}}(Y,Y)=H(Y,Y)$ and the vertical envy $V^{\mathbf{e}}(Y,Y)=V(Y,Y)$.



(a) $\mathbf{N}(Y,Y) \subset \mathbf{N}^e(Y,Y)$.   *(b)* $\mathbf{N}^e(Y,Y) \subset \mathbf{N}(Y,Y)$.

Fig. 3.1: The geometry of envy pure Nash domain $\mathbf{N}^e(Y,Y)$.

Substituting the thresholds $H(Y,Y)$, $H^{\mathbf{e}}(Y,Y)$, $V(Y,Y)$ and $V^{\mathbf{e}}(Y,Y)$.

$$H(Y,Y) = H(Y,Y) + \beta_1 \left[\frac{\alpha_{22}^N - \alpha_{21}^N + \beta_2(\alpha_{11}^N - \alpha_{12}^N)}{1-\beta_1\beta_2}\right]$$

and

$$V(Y,Y) = V(Y,Y) + \beta_2 \left[\frac{\beta_1(\alpha_{22}^N - \alpha_{21}^N) + \alpha_{11}^N - \alpha_{12}^N}{1-\beta_1\beta_2}\right].$$

Rearrange the above inequality, we get

$$0 = \alpha_{22}^N - \alpha_{21}^N + \beta_2(\alpha_{11}^N - \alpha_{12}^N),$$

and

$$0 = \beta_1(\alpha_{22}^N - \alpha_{21}^N) + \alpha_{11}^N - \alpha_{12}^N.$$

(ii) $\mathbf{N}^e(Y,Y) \subset \mathbf{N}(Y,Y)$ if and only if

$$\beta_1(\alpha_{22}^N - \alpha_{21}^N) > \alpha_{12}^N - \alpha_{11}^N \quad \text{and} \quad \beta_2(\alpha_{11}^N - \alpha_{12}^N) > \alpha_{21}^N - \alpha_{22}^N.$$

To clarify item (ii), we use the horizontal envy $H^{\mathbf{e}}(Y,Y) > H(Y,Y)$ and the vertical envy $V^{\mathbf{e}}(Y,Y) > V(Y,Y)$. Substituting the thresholds $H(Y,Y)$, $H^{\mathbf{e}}(Y,Y)$, $V(Y,Y)$ and $V^{\mathbf{e}}(Y,Y)$.

$$H(Y,Y) < H(Y,Y) + \beta_1 \left[\frac{\alpha_{22}^N - \alpha_{21}^N + \beta_2(\alpha_{11}^N - \alpha_{12}^N)}{1-\beta_1\beta_2}\right]$$

and

$$V(Y,Y) < V(Y,Y) + \beta_2 \left[\frac{\beta_1(\alpha_{22}^N - \alpha_{21}^N) + \alpha_{11}^N - \alpha_{12}^N}{1-\beta_1\beta_2}\right].$$

Rearrange the above inequality, we get

$$0<\alpha_{22}^{N}-\alpha_{21}^{N}+\beta_2(\alpha_{11}^{N}-\alpha_{12}^{N}),$$

and

$$0<\beta_1(\alpha_{22}^{N}-\alpha_{21}^{N})+\alpha_{11}^{N}-\alpha_{12}^{N},$$

so, we get

$$\beta_1(\alpha_{22}^{N}-\alpha_{21}^{N})>\alpha_{12}^{N}-\alpha_{11}^{N} \quad \text{and} \quad \beta_2(\alpha_{11}^{N}-\alpha_{12}^{N})>\alpha_{21}^{N}-\alpha_{22}^{N}.$$

(iii) The Nash domains $\mathbf{N}^e(Y,Y)$ and $\mathbf{N}(Y,Y)$ overlaps in the otherwise cases.

Lemma 3.3. *Given a strategy $S\in\boldsymbol{S}$. If $S=(Y,N)$ is a NE, then it is ENE iff*

$$\beta_1(\alpha_{22}^{Y}+\alpha_{21}^{N})<\alpha_{11}^{N}+\alpha_{12}^{Y} \quad \textit{and} \quad \beta_2(\alpha_{11}^{N}+\alpha_{12}^{Y})<\alpha_{22}^{Y}+\alpha_{21}^{N}.$$

Proof. The proof follows from the definitions of the envy Nash region $\mathbf{N^e}(Y,N)$ given in (3.7) and the Nash region $\mathbf{N}(Y,N)$ given in (2.19) when $\mathbf{N}(Y,N)\subset\mathbf{N^e}(Y,N)$.

$$H(Y,N)>H^e(Y,N) \qquad \text{and} \qquad V(Y,N)<V^e(Y,N).$$

Substituting the thresholds horizontal and vertical strategy

$$\begin{aligned} H(Y,N) &> H(Y,N)+\beta_1\left[\frac{\beta_2(\alpha_{11}^{N}+\alpha_{12}^{Y})-(\alpha_{22}^{Y}+\alpha_{21}^{N})}{1-\beta_1\beta_2}\right] \\ V(Y,N) &< V(Y,N)+\beta_2\left[\frac{\alpha_{11}^{N}+\alpha_{12}^{Y}-\beta_1(\alpha_{22}^{Y}+\alpha_{21}^{N})}{1-\beta_1\beta_2}\right]. \end{aligned}$$

Ordering the last inequalities to get

$$\begin{aligned} 0 &> \beta_2(\alpha_{11}^{N}+\alpha_{12}^{Y})-(\alpha_{22}^{Y}+\alpha_{21}^{N}) \\ 0 &< \alpha_{11}^{N}+\alpha_{12}^{Y}-\beta_1(\alpha_{22}^{Y}+\alpha_{21}^{N}). \end{aligned}$$

Simplifies the terms in the previous inequalities, we get

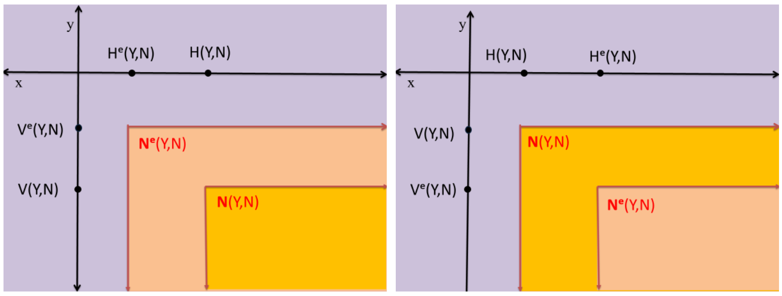
$$\beta_1(\alpha_{22}^{Y}+\alpha_{21}^{N})<\alpha_{11}^{N}+\alpha_{12}^{Y} \quad \text{and} \quad \beta_2(\alpha_{11}^{N}+\alpha_{12}^{Y})<\alpha_{22}^{Y}+\alpha_{21}^{N}.$$

□

Hence, if $\alpha_{11}^N + \alpha_{12}^Y < 0$ and $\alpha_{22}^Y + \alpha_{21}^N < 0$, then $\mathbf{N}(Y,N) \subset \mathbf{N}^e(Y,N)$ holds (see Figure 3.2a) and the following inequalities hold

$$\beta_1 > \frac{\alpha_{12}^Y + \alpha_{11}^N}{\alpha_{22}^Y + \alpha_{21}^N} > 0 \quad \text{and} \quad \beta_2 > \frac{\alpha_{22}^Y + \alpha_{21}^N}{\alpha_{12}^Y + \alpha_{11}^N} > 0.$$

On the other hand, if $\alpha_{11}^N + \alpha_{12}^Y > 0$ and $\alpha_{22}^Y + \alpha_{21}^N > 0$, then $\mathbf{N}^e(Y,N) \subset \mathbf{N}(Y,N)$ (see figure 3.2b).



(a) $\mathbf{N}(Y,N) \subset \mathbf{N}^e(Y,N)$. *(b)* $\mathbf{N}^e(Y,N) \subset \mathbf{N}(Y,N)$.

Fig. 3.2: The geometry of envy pure Nash domain $\mathbf{N}^e(Y,N)$.

We remark that, Lemma 3.3 provides some properties for the Nash domains $\mathbf{N}^e(Y,N)$ and $\mathbf{N}(Y,N)$:

(i) $\mathbf{N}^e(Y,N) = \mathbf{N}(Y,N)$ if $\alpha_{11}^N = -\alpha_{12}^Y$ and $\alpha_{22}^Y = -\alpha_{21}^N$, which means the equilibria coincide.

To clarify item (i), we use the definitions of the envy Nash region $\mathbf{N^e}(Y,N)$ given in (3.7) and the Nash domain $\mathbf{N}(Y,N)$ given in (2.19) when $\mathbf{N}(Y,N) = \mathbf{N^e}(Y,N)$.

$$H(Y,N) = H^e(Y,N) \qquad \text{and} \qquad V(Y,N) = V^e(Y,N).$$

Substituting the thresholds horizontal and vertical strategy

$$\begin{aligned} H(Y,N) &= H(Y,N) + \beta_1 \left[\frac{\beta_2(\alpha_{11}^N + \alpha_{12}^Y) - (\alpha_{22}^Y + \alpha_{21}^N)}{1 - \beta_1\beta_2} \right] \\ V(Y,N) &= V(Y,N) + \beta_2 \left[\frac{\alpha_{11}^N + \alpha_{12}^Y - \beta_1(\alpha_{22}^Y + \alpha_{21}^N)}{1 - \beta_1\beta_2} \right]. \end{aligned}$$

Rearrange the above inequality, we get

$$\begin{aligned} 0 &= \beta_2(\alpha_{11}^N + \alpha_{12}^Y) - (\alpha_{22}^Y + \alpha_{21}^N) \\ 0 &= \alpha_{11}^N + \alpha_{12}^Y - \beta_1(\alpha_{22}^Y + \alpha_{21}^N). \end{aligned}$$

(ii) $\mathbf{N}^e(Y,N) \subset \mathbf{N}(Y,N)$ if and only if

$$\beta_1(\alpha_{22}^Y + \alpha_{21}^N) > \alpha_{12}^Y + \alpha_{11}^N \quad \text{and} \quad \beta_2(\alpha_{11}^N + \alpha_{12}^Y) > \alpha_{21}^N + \alpha_{22}^Y.$$

To clarify item (ii), we use the definitions of the envy Nash domain $\mathbf{N^e}(Y,N)$ given in (3.7) and the Nash domain $\mathbf{N}(Y,N)$ given in (2.19) when $\mathbf{N^e}(Y,N) \subset \mathbf{N}(Y,N)$.

$$H(Y,N) < H^e(Y,N) \qquad \text{and} \qquad V(Y,N) > V^e(Y,N).$$

Substituting the thresholds horizontal and vertical strategy

$$\begin{aligned} H(Y,N) &< H(Y,N) + \beta_1 \left[\frac{\beta_2(\alpha_{11}^N + \alpha_{12}^Y) - (\alpha_{22}^Y + \alpha_{21}^N)}{1 - \beta_1\beta_2} \right] \\ V(Y,N) &> V(Y,N) + \beta_2 \left[\frac{\alpha_{11}^N + \alpha_{12}^Y - \beta_1(\alpha_{22}^Y + \alpha_{21}^N)}{1 - \beta_1\beta_2} \right]. \end{aligned}$$

Rearrange the above inequality, we get

$$\begin{aligned} 0 &< \beta_2(\alpha_{11}^N + \alpha_{12}^Y) - (\alpha_{22}^Y + \alpha_{21}^N) \\ 0 &> \alpha_{11}^N + \alpha_{12}^Y - \beta_1(\alpha_{22}^Y + \alpha_{21}^N). \end{aligned}$$

Rearranging the terms in the previous inequalities, we get

$$\beta_1(\alpha_{22}^Y + \alpha_{21}^N) > \alpha_{12}^Y + \alpha_{11}^N \quad \text{and} \quad \beta_2(\alpha_{11}^N + \alpha_{12}^Y) > \alpha_{21}^N + \alpha_{22}^Y.$$

(iii) The Nash domains $\mathbf{N}^e(Y,N)$ and $\mathbf{N}(Y,N)$ overlaps in the otherwise cases.

Lemma 3.4. *Given a strategy $S \in \boldsymbol{S}$. If $S = (N,Y)$ is NE, then it is an envy Nash Equilibrium if and only if*

$$\beta_1(\alpha_{22}^N + \alpha_{21}^Y) < \alpha_{11}^Y + \alpha_{12}^N \quad \textit{and} \quad \beta_2(\alpha_{11}^Y + \alpha_{12}^N) < \alpha_{22}^N + \alpha_{21}^Y.$$

Proof. The proof is straight forward from the definitions of the envy Nash region $\mathbf{N^e}(N,Y)$ given in (3.9) and the Nash region $\mathbf{N}(N,Y)$ given in (2.24) when $\mathbf{N}(N,Y) \subset \mathbf{N^e}(N,Y)$.

$$H(N,Y) < H^e(N,Y) \qquad \text{and} \qquad V(N,Y) > V^e(N,Y).$$

Substituting the thresholds horizontal and vertical strategy

$$\begin{aligned} H(N,Y) &< H(N,Y)+\beta_1\left[\frac{\alpha^N_{22}+\alpha^Y_{21}-\beta_2(\alpha^Y_{11}+\alpha^N_{12})}{1-\beta_1\beta_2}\right] \\ V(N,Y) &> V(N,Y)+\beta_2\left[\frac{\beta_1(\alpha^N_{22}+\alpha^Y_{21})-(\alpha^Y_{11}+\alpha^N_{12})}{1-\beta_1\beta_2}\right]. \end{aligned}$$

Rearrange the above inequality, we get

$$\begin{aligned} 0 &< \alpha^N_{22}+\alpha^Y_{21}-\beta_2(\alpha^Y_{11}+\alpha^N_{12}) \\ 0 &> \beta_1(\alpha^N_{22}+\alpha^Y_{21})-(\alpha^Y_{11}+\alpha^N_{12}). \end{aligned}$$

Rearranging the terms in the previous inequalities, we get

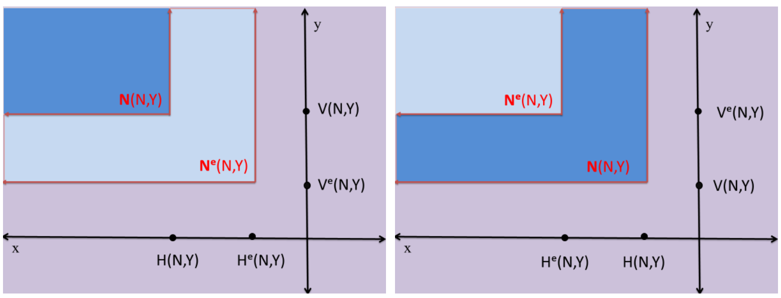
$$\beta_1(\alpha^N_{22}+\alpha^Y_{21})<\alpha^Y_{11}+\alpha^N_{12} \quad \text{and} \quad \beta_2(\alpha^Y_{11}+\alpha^N_{12})<\alpha^N_{22}+\alpha^Y_{21}.$$

□

Hence, if $\alpha^Y_{11}+\alpha^N_{12}<0$ and $\alpha^N_{22}+\alpha^Y_{21}<0$, then $\mathbf{N}(N,Y)\subset\mathbf{N}^e(N,Y)$ holds (see Figure 3.3a) and the following inequalities hold

$$\beta_1>\frac{\alpha^N_{12}+\alpha^Y_{11}}{\alpha^N_{22}+\alpha^Y_{21}}>0 \quad \text{and} \quad \beta_2>\frac{\alpha^N_{22}+\alpha^Y_{21}}{\alpha^N_{12}+\alpha^Y_{11}}>0.$$

On the other hand, if $\alpha^Y_{11}+\alpha^N_{12}>0$ and $\alpha^N_{22}+\alpha^Y_{21}>0$, then $\mathbf{N}^e(N,Y)\subset\mathbf{N}(N,Y)$ (see figure 3.3b). We remark that, Lemma 3.2 provides some



(a) $\mathbf{N}(N,Y)\subset\mathbf{N}^e(N,Y)$. (b) $\mathbf{N}^e(N,Y)\subset\mathbf{N}(N,Y)$.

Fig. 3.3: The geometry of envy pure Nash domain $\mathbf{N}^e(N,Y)$.

properties for the Nash domains $\mathbf{N}^e(N,Y)$ and $\mathbf{N}(N,Y)$:

(i) $\mathbf{N}^e(N,Y) = \mathbf{N}(N,Y)$ if $\alpha_{11}^Y = -\alpha_{12}^N$ and $\alpha_{22}^N = -\alpha_{21}^Y$, which means the equilibria coincide. This means that the equilibria coincide $\beta_1\beta_2 = 1$ assuming $\alpha_{11}^Y + \alpha_{12}^N \neq 0$ and $\alpha_{22}^N + \alpha_{21}^Y \neq 0$. To clarify item (i), we use the definitions of the envy Nash domain $\mathbf{N^e}(N,Y)$ given in (3.9) and the Nash region $\mathbf{N}(N,Y)$ given in (2.24) when $\mathbf{N}(N,Y) = \mathbf{N^e}(N,Y)$.

$$H(N,Y) = H^e(N,Y) \qquad \text{and} \qquad V(N,Y) = V^e(N,Y).$$

Substituting the thresholds horizontal and vertical strategy

$$\begin{aligned} H(N,Y) &= H(N,Y) + \beta_1\left[\frac{\alpha_{22}^N + \alpha_{21}^Y - \beta_2(\alpha_{11}^Y + \alpha_{12}^N)}{1-\beta_1\beta_2}\right] \\ V(N,Y) &= V(N,Y) + \beta_2\left[\frac{\beta_1(\alpha_{22}^N + \alpha_{21}^Y) - (\alpha_{11}^Y + \alpha_{12}^N)}{1-\beta_1\beta_2}\right]. \end{aligned}$$

Simplifies the terms in the previous inequalities, we get

$$\begin{aligned} 0 &= \alpha_{22}^N + \alpha_{21}^Y - \beta_2(\alpha_{11}^Y + \alpha_{12}^N) \\ 0 &= \beta_1(\alpha_{22}^N + \alpha_{21}^Y) - (\alpha_{11}^Y + \alpha_{12}^N). \end{aligned}$$

(ii) $\mathbf{N}^e(N,Y) \subset \mathbf{N}(N,Y)$ if and only if

$$\beta_1(\alpha_{22}^N + \alpha_{21}^Y) > \alpha_{12}^N + \alpha_{11}^Y \quad \text{and} \quad \beta_2(\alpha_{11}^Y + \alpha_{12}^N) > \alpha_{21}^Y + \alpha_{22}^N.$$

To clarify item (ii), we use the definitions of the envy Nash domain $\mathbf{N^e}(N,Y)$ given in (3.9) and the Nash region $\mathbf{N}(N,Y)$ given in (2.24) when $\mathbf{N^e}(N,Y) \subset \mathbf{N}(N,Y)$.

$$H(N,Y) > H^e(N,Y) \qquad \text{and} \qquad V(N,Y) < V^e(N,Y).$$

Substituting the thresholds horizontal and vertical strategy

$$\begin{aligned} H(N,Y) &> H(N,Y) + \beta_1\left[\frac{\alpha_{22}^N + \alpha_{21}^Y - \beta_2(\alpha_{11}^Y + \alpha_{12}^N)}{1-\beta_1\beta_2}\right] \\ V(N,Y) &< V(N,Y) + \beta_2\left[\frac{\beta_1(\alpha_{22}^N + \alpha_{21}^Y) - (\alpha_{11}^Y + \alpha_{12}^N)}{1-\beta_1\beta_2}\right]. \end{aligned}$$

Rearranging the terms in the previous inequalities, we get

$$\begin{aligned} 0 &> \alpha_{22}^N + \alpha_{21}^Y - \beta_2(\alpha_{11}^Y + \alpha_{12}^N) \\ 0 &< \beta_1(\alpha_{22}^N + \alpha_{21}^Y) - (\alpha_{11}^Y + \alpha_{12}^N). \end{aligned}$$

Simplifies the terms in the previous inequalities, we get

$$\beta_1(\alpha_{22}^N + \alpha_{21}^Y) > \alpha_{12}^N + \alpha_{11}^Y \quad \text{and} \quad \beta_2(\alpha_{11}^Y + \alpha_{12}^N) > \alpha_{21}^Y + \alpha_{22}^N.$$

(iii) the Nash domains $\mathbf{N}^e(N,Y)$ and $\mathbf{N}(N,Y)$ overlaps in the otherwise cases.

Lemma 3.5. *Given a strategy $S \in \boldsymbol{S}$. If $S = (N,N)$ is NE, then it is ENE iff*

$$\beta_1(\alpha_{22}^Y - \alpha_{21}^Y) < \alpha_{12}^Y - \alpha_{11}^Y \quad \textit{and} \quad \beta_2(\alpha_{11}^Y - \alpha_{12}^Y) < \alpha_{21}^Y - \alpha_{22}^Y.$$

Proof. The proof follows from the definitions of the envy Nash domain $\mathbf{N^e}(N,N)$ given in (3.11) and the Nash region $\mathbf{N}(N,N)$ given in (2.28) when $\mathbf{N}(N,N) \subset \mathbf{N^e}(N,N)$.

$$H(N,N) < H^e(N,N) \qquad \text{and} \qquad V(N,N) < V^e(N,N).$$

Substituting the thresholds horizontal and vertical strategy

$$\begin{aligned} H(N,N) &< H(N,N) + \beta_1 \left[\frac{\alpha_{21}^Y - \alpha_{22}^Y + \beta_2(\alpha_{12}^Y - \alpha_{11}^Y)}{1 - \beta_1\beta_2} \right] \\ V(N,N) &< V(N,N) + \beta_2 \left[\frac{\beta_1(\alpha_{21}^Y - \alpha_{22}^Y) + \alpha_{12}^Y - \alpha_{11}^Y}{1 - \beta_1\beta_2} \right]. \end{aligned}$$

Rearranging the terms in the previous inequalities, we get

$$\begin{aligned} 0 &< \alpha_{21}^Y - \alpha_{22}^Y + \beta_2(\alpha_{12}^Y - \alpha_{11}^Y) \\ 0 &< \beta_1(\alpha_{21}^Y - \alpha_{22}^Y) + \alpha_{12}^Y - \alpha_{11}^Y. \end{aligned}$$

Rearrange the above inequality, we get

$$\beta_1(\alpha_{22}^Y - \alpha_{21}^Y) < \alpha_{12}^Y - \alpha_{11}^Y \quad \text{and} \quad \beta_2(\alpha_{11}^Y - \alpha_{12}^Y) < \alpha_{21}^Y - \alpha_{22}^Y.$$

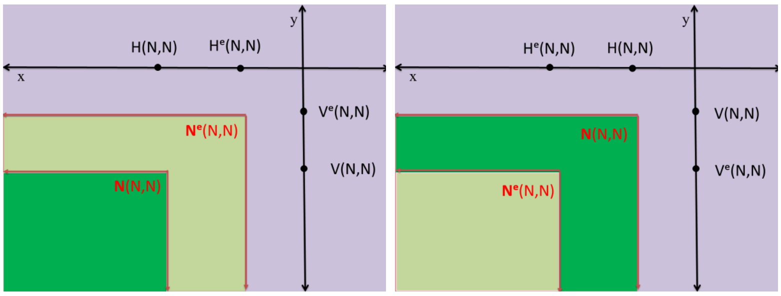
□

Hence, if players with type t_1(resp. t_2) like more being with players with type t_2(resp. t_1) than being together making decision Y (means

$\alpha^Y_{11}{<}\alpha^Y_{12}$ and $\alpha^Y_{22}{<}\alpha^Y_{21}$), then $\mathbf{N}(N,N) \subset \mathbf{N}^e(N,N)$ holds (see Figure 3.4a) and the following inequalities hold

$$\beta_1{>}0{>}\frac{\alpha^Y_{12}-\alpha^Y_{11}}{\alpha^Y_{22}-\alpha^Y_{21}} \quad \text{and} \quad \beta_2{>}0{>}\frac{\alpha^Y_{22}-\alpha^Y_{21}}{\alpha^Y_{12}-\alpha^Y_{11}}.$$

On the other hand, if players with type t_1(resp. t_2) like more being together than being with players with type t_2(resp. t_1) making decision Y (means $\alpha^Y_{11}{>}\alpha^Y_{12}$ and $\alpha^Y_{22}{>}\alpha^Y_{21}$), then $\mathbf{N}^e(N,N) \subset \mathbf{N}(N,N)$ (see figure 3.4b).



(a) $\mathbf{N}(N,N) \subset \mathbf{N}^e(N,N)$. (b) $\mathbf{N}^e(N,N) \subset \mathbf{N}(N,N)$.

Fig. 3.4: The geometry of envy pure Nash domain $\mathbf{N}^e(N,N)$.

We remark that, Lemma 3.2 provides some properties for the Nash domains $\mathbf{N}^e(N,N)$ and $\mathbf{N}(N,N)$:

(i) $\mathbf{N}^e(N,N) = \mathbf{N}(N,N)$ if $\alpha^Y_{11} = \alpha^Y_{12}$ and $\alpha^Y_{22} = \alpha^Y_{21}$, which means that the equilibria coincide.

To clarify item (i), we use the definitions of the envy Nash domain $\mathbf{N^e}(N,N)$ given in (3.11) and the Nash region $\mathbf{N}(N,N)$ given in (2.28) when $\mathbf{N}(N,N) = \mathbf{N^e}(N,N)$.

$$H(N,N) = H^e(N,N) \qquad \text{and} \qquad V(N,N) = V^e(N,N).$$

Substituting the thresholds horizontal and vertical strategy

$$\begin{aligned} H(N,N) &= H(N,N) + \beta_1\left[\frac{\alpha^Y_{21}-\alpha^Y_{22}+\beta_2(\alpha^Y_{12}-\alpha^Y_{11})}{1-\beta_1\beta_2}\right] \\ V(N,N) &= V(N,N) + \beta_2\left[\frac{\beta_1(\alpha^Y_{21}-\alpha^Y_{22})+\alpha^Y_{12}-\alpha^Y_{11}}{1-\beta_1\beta_2}\right]. \end{aligned}$$

Ordering the last inequalities to get

$$\begin{aligned} 0 &= \alpha^Y_{21} - \alpha^Y_{22} + \beta_2(\alpha^Y_{12} - \alpha^Y_{11}) \\ 0 &= \beta_1(\alpha^Y_{21} - \alpha^Y_{22}) + \alpha^Y_{12} - \alpha^Y_{11}. \end{aligned}$$

(ii) $\mathbf{N}^e(N,N) \subset \mathbf{N}(N,N)$ if and only if

$$\beta_1(\alpha^Y_{22} - \alpha^Y_{21}) > \alpha^Y_{12} - \alpha^Y_{11} \quad \text{and} \quad \beta_2(\alpha^Y_{11} - \alpha^Y_{12}) > \alpha^Y_{21} - \alpha^Y_{22}.$$

To clarify item (ii), we use the definitions of the envy Nash domain $\mathbf{N^e}(N,N)$ given in (3.11) and the Nash domain $\mathbf{N}(N,N)$ given in (2.28) when $\mathbf{N}^e(N,N) \subset \mathbf{N}(N,N)$.

$$H(N,N) > H^e(N,N) \qquad \text{and} \qquad V(N,N) > V^e(N,N).$$

Substituting the thresholds horizontal and vertical strategy

$$\begin{aligned} H(N,N) &> H(N,N) + \beta_1 \left[\frac{\alpha^Y_{21} - \alpha^Y_{22} + \beta_2(\alpha^Y_{12} - \alpha^Y_{11})}{1 - \beta_1\beta_2} \right] \\ V(N,N) &> V(N,N) + \beta_2 \left[\frac{\beta_1(\alpha^Y_{21} - \alpha^Y_{22}) + \alpha^Y_{12} - \alpha^Y_{11}}{1 - \beta_1\beta_2} \right]. \end{aligned}$$

Rearranging the terms in the previous inequalities, we get

$$\begin{aligned} 0 &> \alpha^Y_{21} - \alpha^Y_{22} + \beta_2(\alpha^Y_{12} - \alpha^Y_{11}) \\ 0 &> \beta_1(\alpha^Y_{21} - \alpha^Y_{22}) + \alpha^Y_{12} - \alpha^Y_{11}. \end{aligned}$$

Simplifies the terms in the previous inequalities, we get

$$\beta_1(\alpha^Y_{22} - \alpha^Y_{21}) > \alpha^Y_{12} - \alpha^Y_{11} \quad \text{and} \quad \beta_2(\alpha^Y_{11} - \alpha^Y_{12}) > \alpha^Y_{21} - \alpha^Y_{22}.$$

(iii) the Nash domains $\mathbf{N}^e(N,N)$ and $\mathbf{N}(N,N)$ overlaps in the otherwise cases.

3.2 *Geometric classes of envy tilings*

The representation of the Nash domains

$$\mathbf{N}(Y,Y), \mathbf{N}(Y,N), \mathbf{N}(N,Y), \mathbf{N}(N,N)$$

and the envy Nash domains

$$\mathbf{N}^e(Y,Y), \mathbf{N}^e(Y,N), \mathbf{N}^e(N,Y), \mathbf{N}^e(N,N)$$

in the Cartesian xy-plan determine an envy decision tiling. These tilings characterize geometrically all *envy Nash equilibria.*

Recall the definition of the entries of the crowding matrix A as defined in the Definition 2.10. We now introduce another matrix called **balanced threshold weight matrix**. The coordinates of these matrices have a main influence on the order of the horizontal and vertical thresholds.

Definition 3.3. *Let B be the balanced threshold weight matrix whose coordinates are given by*

$$\begin{aligned} B &= \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \\ &= \begin{pmatrix} A_{11}(n_1-1) - A_{12}n_2 & A_{11}(n_1-1) - A_{12}n_2 \\ A_{22}(n_2-1) - A_{21}n_1 & A_{22}(n_2-1) - A_{21}n_1 \end{pmatrix}. \end{aligned}$$

The signs of the coordinates of the influence matrix and balanced threshold weight matrix determine a certain order for the horizontal and vertical strategic thresholds.

Definition 3.4. *An envy tiling is structurally stable if its horizontal and vertical cut points do not intersect.*

Definition 3.5. *An envy tiling has bifurcation if least two of its horizontal or vertical cut points intersect.*

Definition 3.6. *Two envy tilings are combinatorial equivalent if the orders of the horizontal and vertical cut points are equal in both tilings.*

We call the pair of horizontal and vertical braids the envy human decision chromosomes as they play a central role to determine the human decision behavior, see Figure 3.5 where we show only the horizontal braid of envy human decision chromosomes for players with type

t_1 (the vertical braid of envy human decision chromosomes for players with type t_2 follows similarly to Figure 3.5). Each pair of lines transversal to the horizontal and vertical braids, respectively, determines a unique envy decision tiling. The number of permutations for ordering *the horizontal thresholds* along the x-axis is huge (8! without horizontal bifurcations) and same number of permutations for the ordering the vertical thresholds along y-axis without vertical bifurcations. However, we will focus on a certain order for the horizontal (resp. vertical) thresholds presented in Figure 3.5 where there are $1024 = 32 \times 32$ combinatorial classes of envy decision tilings, and $256 = 16 \times 16$ of them are being structurally stable and $768 = 1024 - 256$ combinatorial classes of bifurcation decision tilings.

In Figure 3.5, note that:

- pink circles • represent the horizontal envy threshold $H^e(N, N)$,
- black circles • represent the horizontal envy threshold $H^e(N, Y)$,
- green circles • represent *the horizontal threshold* $H(N, N)$,
- blue circles • represent *the horizontal threshold* $H(N, Y)$,
- orange circles • represent *the horizontal threshold* $H(Y, N)$,
- red circles • represent *the horizontal threshold* $H(Y, Y)$,
- yellow circles • represent the horizontal envy threshold $H^e(Y, N)$,
- gray circles • represent the horizontal envy threshold $H^e(Y, Y)$,
- light green arrows $\leftrightarrow$ represent the occurrence of four times of horizontal (resp. vertically) bifurcations,
- and light orange arrows $\leftrightarrow$ represent the occurrence of three times of horizontal (resp. vertical) bifurcations.

In Figures 3.6, 3.7 and 3.8, we present three envy decisions tilings where

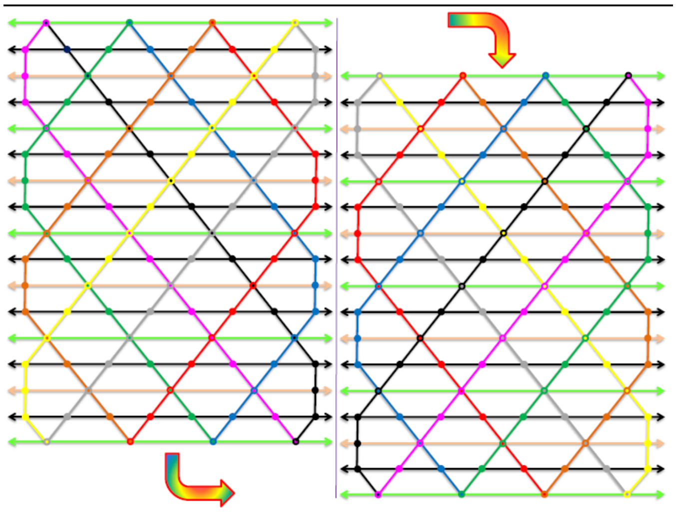


Fig. 3.5: Horizontal braid of envy human decision chromosomes for players with type t_1.

- regions with cohesive uniqueness Nash equilibria domain

$$\begin{array}{ll} U(Y,Y) \subset \mathbf{N}(Y,Y) & \text{colored red,} \\ U(Y,N) \subset \mathbf{N}(Y,N) & \text{colored orange,} \\ U(N,Y) \subset \mathbf{N}(N,Y) & \text{colored blue ,} \\ U(N,N) \subset \mathbf{N}(N,N) & \text{colored green;} \end{array}$$

- regions with cohesive uniqueness envy Nash equilibria domains

$$\begin{array}{ll} U^e(Y,Y) \subset N^e(Y,Y) & \text{colored light red,} \\ U^e(Y,N) \subset N^e(Y,N) & \text{colored light orange,} \\ U^e(N,Y) \subset N^e(N,Y) & \text{colored light blue ,} \\ U^e(N,N) \subset N^e(N,N) & \text{colored light green;} \end{array}$$

- regions with neither cohesive Nash equilibria nor envy Nash equilibria colored purple;
- regions with two cohesive Nash equilibria colored yellow,
- regions with three cohesive Nash equilibria colored brown,
- regions with four cohesive Nash equilibria colored pink,

- regions with five cohesive Nash equilibria colored light yellow,
- regions with six cohesive Nash equilibria colored gray,
- regions with seven cohesive Nash equilibria colored chartreuse,
- regions with eight cohesive Nash equilibria colored rainbow.

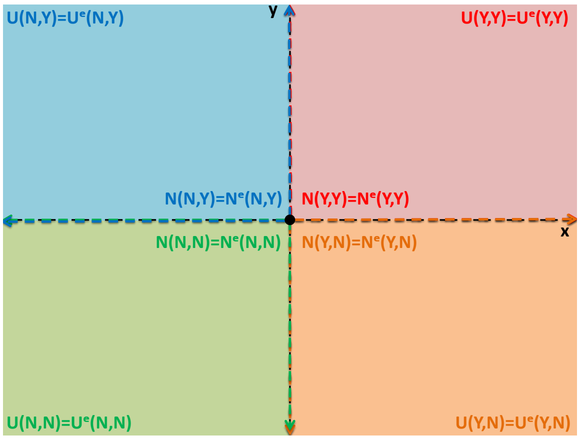


Fig. 3.6: Envy Nash equilibria domains when $A_{11} = A_{12} = A_{21} = A_{22} = 0$.

In Figure 3.6, for every taste x and y, there is only one cohesive NE and one envy NE, except along the horizontal and vertical axes where there are two cohesive NE and two envy NE, and at the origin where there are four cohesive NE and four envy NE.

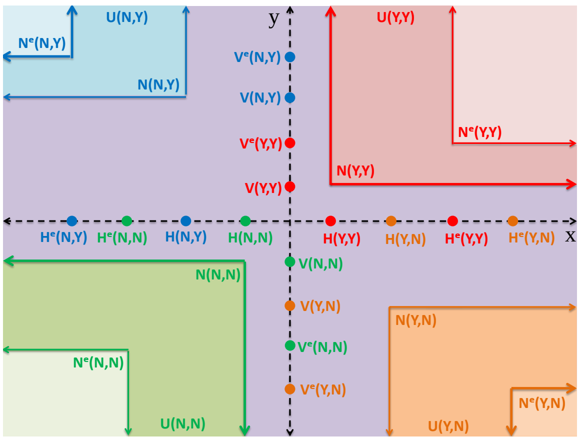


Fig. 3.7: Simple strategy of cohesive envy Nash equilibria domains.

In Figure 3.7, we show one possible tiling with simple strategy of the cohesive envy Nash equilibria domains when

$$\alpha_{11}^{d}>\alpha_{12}^{d},\quad \alpha_{22}^{d}>\alpha_{21}^{d},\quad \alpha_{11}^{d}+\alpha_{12}^{d'}>0,\quad \alpha_{22}^{d}+\alpha_{21}^{d'}>0 \quad \text{and} \quad d\neq d'\in\{Y,N\}$$

and

$$A_{11}<0,\quad A_{22}<0,\quad A_{12}>0,\quad A_{21}>0 \quad \text{and} \quad B_{12}<0,\quad B_{21}<0.$$

We show that there is an unbounded region colored purple with neither cohesive Nash equilibrium nor *cohesive envy Nash equilibrium*.

In Figure 3.8, we show the high complexity of distributing the cohesive envy Nash equilibria domains when

$$\alpha_{11}^{d}<\alpha_{12}^{d},\quad \alpha_{22}^{d}<\alpha_{21}^{d},\quad \alpha_{11}^{d}+\alpha_{12}^{d'}<0,\quad \alpha_{22}^{d}+\alpha_{21}^{d'}<0 \quad \text{and} \quad d\neq d'\in\{Y,N\}$$

and

$$A_{11}>0,\quad A_{22}>0,\quad A_{12}<0,\quad A_{21}<0 \quad \text{and} \quad B_{12}>0,\quad B_{21}>0.$$

We show that there are regions with two, three, four, five, six, seven and eight cohesive Nash equilibria colored yellow, brown, pink, light yellow, gray, chartreuse and rainbow, respectively.

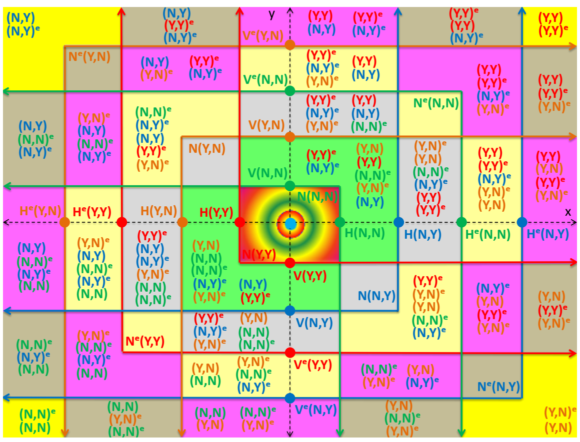


Fig. 3.8: The complexity of envy Nash equilibria domain.

3.3 Disparate envy Nash equilibria

Here we study the disparate *envy NE.*

Definition 3.7. *The strategic envy set* (l_1, l_2) *is the set of all pure strategies* $S \in \boldsymbol{S}$: $l_1(S) = l_1$ *and* $l_2(S) = l_2$. *The cohesive strategic envy set* (l_1, l_2) *is the set all pure strategies* $S \in \boldsymbol{S}$ *with* $l_1 = 0$ *or* $l_1 = n_1$ *and* $l_2 = 0$ *or* $l_2 = n_2$. *The disparate strategic envy set* (l_1, l_2) *is the set all pure strategies envy set that are not cohesive strategic envy set.*

Definition 3.8. *The pure envy Nash Equilibrium (set)* (l_1, l_2) *is a strategic envy set whose strategies are NE. The (pure) envy Nash region* $\boldsymbol{N}^{\mathbf{e}}(l_1, l_2)$ *contains all taste pairs* (x, y) *so that* (l_1, l_2) *is a NE set.*

The pure *ENE* set (l_1, l_2) is *cohesive* if $l_1 = 0$ or $l_1 = n_1$ and $l_2 = 0$ or $l_2 = n_2$. Otherwise, the pure *ENE* set (l_1, l_2) is *disparate envy.*

Lemma 3.6. *let* (l_1, l_2) *be an envy Nash Equilibrium set.*

(i) If $A_{11} > \beta_1 A_{21}$, *then* $l_1 \in \{0, n_1\}$

(ii) If $A_{22} > \beta_2 A_{12}$, *then* $l_2 \in \{0, n_2\}$

Furthermore, if $A_{11} > \beta_1 A_{21}$ *and* $A_{22} > \beta_2 A_{12}$ *then* (l_1, l_2) *is cohesive envy Nash Equilibrium.*

Proof. The proof is by contradiction. Assume the envy strategy (l_1, l_2) is a *NE* for $l_1 \in \{1, 2, \ldots, n_1 - 1\}$. So we must have

$$\begin{aligned} U_1(Y; l_1, l_2, \beta_1) &\geq U_1(N; l_1 - 1, l_2, \beta_1) \quad \text{and} \\ U_1(N; l_1, l_2, \beta_1) &\geq U_1(Y; l_1 + 1, l_2, \beta_1). \end{aligned} \tag{3.22}$$

Substituting the envy utility functions (3.1) in the first inequality (3.22), we get

$$\begin{aligned}\omega_1^Y \quad & +\alpha_{11}^Y(l_1-1)+\alpha_{12}^Y l_2-\beta_1(\omega_2^Y+\alpha_{22}^Y(l_2-1)+\alpha_{21}^Y l_1)\geq \\ \omega_1^N \quad & +\alpha_{11}^N(n_1-(l_1-1)-1)+\alpha_{12}^N(n_2-l_2) \\ & -\beta_1(\omega_2^N+\alpha_{22}^N(n_2-l_2-1)+\alpha_{21}^N(n_1-(l_1-1))),\end{aligned}$$

and substituting the envy utility functions (3.1) in the second inequality (3.22), we get

$$\begin{aligned}\omega_1^N \quad & +\alpha_{11}^N(n_1-l_1-1)+\alpha_{12}^N(n_2-l_2) \\ & -\beta_1(\omega_2^N+\alpha_{22}^N(n_2-l_2-1)+\alpha_{21}^N(n_1-l_1))\geq \\ \omega_1^Y \quad & +\alpha_{11}^Y(l_1+1-1)+\alpha_{12}^Y l_2-\beta_1(\omega_2^Y+\alpha_{22}^Y(l_2-1)+\alpha_{21}^Y(l_1+1)).\end{aligned}$$

Ordering the last inequalities to get

$$\begin{aligned}\alpha_{11}^Y(l_1-1) \quad & +\alpha_{11}^N(n_1-l_1-1)-\alpha_{11}^Y(l_1)-\alpha_{11}^N(n_1-l_1)\geq \\ \beta_1(\alpha_{21}^N(n_1-l_1)) \quad & +\beta_1(\alpha_{21}^Y l_1)-\beta_1(\alpha_{21}^Y(l_1+1))+\beta_1(\alpha_{21}^N(n_1-l_1+1)).\end{aligned}$$

That is

$$A_{11}\leq \beta_1 A_{21},$$

which contradicts that

$$A_{11}>\beta_1 A_{21}.$$

Hence, Lemma 3.6(i) is done. We prove in same way the other items.

□

3.4 *Special case of Theorem 3.1*

In this section we study a special case of Theorem 3.1 when the envy parameters $\beta_1 = 1$ and $\beta_2 = 1$.

To start this special case we first simplify the envy utility functions.

The envy utility $U_1^{\mathbf{e}} : \mathbf{D}\times\mathbf{O}\times\mathbb{R}^+ \longrightarrow \mathbb{R}$ of an envy player whose type is t_1 and makes decision Y is given by

$$\begin{aligned}U_1^{\mathbf{e}}(Y;l_1,l_2,1) &= U_1(Y;l_1,l_2)-U_2(Y;l_1,l_2) \qquad (3.23)\\ &= (\omega_1^Y-\omega_2^Y)+\alpha_{11}^Y(l_1-1)+\alpha_{12}^Y l_2-\alpha_{22}^Y(l_2-1)-\alpha_{21}^Y l_1.\end{aligned}$$

The envy utility $U_1^{\mathbf{e}} : \mathbf{D} \times \mathbf{O} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ of an envy player whose type is t_1 and makes decision N is given by

$$\begin{aligned} U_1^{\mathbf{e}}(N; l_1, l_2, 1) &= U_1(N; l_1, l_2) - U_2(N; l_1, l_2) \quad (3.24) \\ &= (\omega_1^N - \omega_2^N) + \alpha_{11}^N(n_1 - l_1 - 1) + \alpha_{12}^N(n_2 - l_2) \\ &\quad -\alpha_{22}^N(n_2 - l_2 - 1) - \alpha_{21}^N(n_1 - l_1). \end{aligned}$$

The envy utility $U_2^{\mathbf{e}} : \mathbf{D} \times \mathbf{O} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ of an envy player whose type is t_2 and makes decision Y is given by

$$\begin{aligned} U_2^{\mathbf{e}}(Y; l_1, l_2, 1) &= U_2(Y; l_1, l_2) - U_1(Y; l_1, l_2) \\ &= (\omega_2^Y - \omega_1^Y) + \alpha_{22}^Y(l_2 - 1) + \alpha_{21}^Y l_1 \\ &\quad -\alpha_{11}^Y(l_1 - 1) - \alpha_{12}^Y l_2 \ . \quad (3.25) \end{aligned}$$

The envy utility $U_2^{\mathbf{e}} : \mathbf{D} \times \mathbf{O} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ of an envy player whose type is t_2 and makes decision N is given by

$$\begin{aligned} U_2^{\mathbf{e}}(N; l_1, l_2, 1) &= U_2(N; l_1, l_2) - U_1(N; l_1, l_2) \\ &= (\omega_2^N - \omega_1^N) + \alpha_{22}^N(n_2 - l_2 - 1) + \alpha_{21}^N(n_1 - l_1) \\ &\quad -\alpha_{11}^N(n_1 - l_1 - 1) - \alpha_{12}^N(n_2 - l_2). \quad (3.26) \end{aligned}$$

Theorem 3.7. *Assume that $\beta_1 = 1$ and $\beta_2 = 1$. The envy cohesive strategy $S_{\mathbf{e}} = (Y, Y)$ is ENE iff $(x, y) \in N_s^e(Y, Y)$, where the envy Nash region $N_s^e(Y, Y)$ is given by*

$$N_s^e(Y, Y) = \{(x, y) \in \mathbb{R}^2 : Z_L^e(Y, Y) \le y - x \le Z_R^e(Y, Y)\},$$

the left envy threshold $Z_L^e(Y, Y)$ is given by

$$Z_L^e(Y, Y) = V(Y, Y) - H(Y, Y) + \alpha_{11}^N - \alpha_{12}^N,$$

the right envy threshold $Z_R^e(Y, Y)$ is given by

$$Z_R^e(Y, Y) = V(Y, Y) - H(Y, Y) - \alpha_{22}^N + \alpha_{21}^N,$$

where the horizontal and vertical cut points are as given in (2.16).

Proof. The cohesive envy strategy $S_{\mathbf{e}} = (Y, Y)$ is NE iff

$$\begin{aligned} U_1^{\mathbf{e}}(Y; n_1, n_2, 1) &\ge U_1^{\mathbf{e}}(N; n_1 - 1, n_2, 1) \quad \text{and} \quad (3.27) \\ U_2^{\mathbf{e}}(Y; n_1, n_2, 1) &\ge U_2^{\mathbf{e}}(N; n_1, n_2 - 1, 1). \end{aligned}$$

Substituting the envy utility functions from (3.23) and (3.24) in the first inequality of (3.27), we obtain

$$\begin{aligned}&(\omega_1^Y-\omega_2^Y)+\alpha_{11}^Y(n_1-1)+\alpha_{12}^Y n_2-\alpha_{22}^Y(n_2-1)-\alpha_{21}^Y n_1 \geq\\&(\omega_1^N-\omega_2^N)+\alpha_{11}^N(n_1-1-(n_1-1))+\alpha_{12}^N(n_2-n_2)\\&-\alpha_{22}^N(n_2-n_2-1)-\alpha_{21}^N(n_1-(n_1-1)).\end{aligned}$$

Rearrange the above inequality, we get

$$(\omega_1^Y-\omega_1^N)-(\omega_2^Y-\omega_2^N) \quad \geq -\alpha_{11}^Y(n_1-1)-\alpha_{12}^Y n_2+\alpha_{22}^Y(n_2-1)+\alpha_{21}^Y n_1+\alpha_{22}^N-\alpha_{21}^N.$$

By simplifying the previous inequality using the definition of horizontal and vertical tastes x and y given in (2.8), we get

$$\begin{aligned}x-y \geq\ & -\alpha_{11}^Y(n_1-1)-\alpha_{12}^Y n_2+\alpha_{22}^Y(n_2-1)\\&+\alpha_{21}^Y n_1+\alpha_{22}^N-\alpha_{21}^N.\end{aligned} \tag{3.28}$$

Substituting *the horizontal and vertical cut points* from (2.16) in the last inequality, we obtain

$$y-x \leq V(Y,Y)-H(Y,Y)-\alpha_{22}^N+\alpha_{21}^N.$$

Hence,

$$y-x \leq Z_R^e(Y,Y), \tag{3.29}$$

where $Z_R^e(Y,Y)$ is as given in the statement of Theorem 3.7.
Similarly, we substitute the envy utility functions from (3.25) and (3.26) in the second inequality of (3.27), we obtain

$$\begin{aligned}&(\omega_2^Y-\omega_1^Y)+\alpha_{22}^Y(n_2-1)+\alpha_{21}^Y n_1-\alpha_{11}^Y(n_1-1)-\alpha_{12}^Y n_2 \geq\\&(\omega_2^N-\omega_1^N)+\alpha_{22}^N(n_2-1-(n_2-1))+\alpha_{21}^N(n_1-n_1)\\&-\alpha_{11}^N(n_1-n_1-1)-\alpha_{12}^N(n_2-(n_2-1)).\end{aligned}$$

Rearrange the above inequality, we get

$$(\omega_2^Y-\omega_2^N)-(\omega_1^Y-\omega_1^N) \geq -\alpha_{22}^Y(n_2-1)-\alpha_{21}^Y n_1+\alpha_{11}^Y(n_1-1)+\alpha_{12}^Y n_2+\alpha_{11}^N-\alpha_{12}^N.$$

By simplifying the previous inequality using the definition of horizontal and vertical tastes x and y given in (2.8), we get

$$\begin{aligned}y-x \geq\ & -\alpha_{22}^Y(n_2-1)-\alpha_{21}^Y n_1+\alpha_{11}^Y(n_1-1)\\&+\alpha_{12}^Y n_2+\alpha_{11}^N-\alpha_{12}^N.\end{aligned} \tag{3.30}$$

Substituting *the horizontal* *and vertical* *cut points* from (2.16) in the last inequality, we obtain

$$y - x \geq V(Y,Y) - H(Y,Y) + \alpha_{11}^{N} - \alpha_{12}^{N}.$$

Hence,

$$x - y \geq Z_L^e(Y,Y), \tag{3.31}$$

where $Z_L^e(Y,Y)$ is as given in the statement of Theorem 3.7. Combining inequalities (3.29) and (3.31), we get

$$Z_L^e(Y,Y) \leq y - x \leq Z_R^e(Y,Y).$$

□

Lemma 3.8. *Assume that $\beta_1 = 1$ and $\beta_2 = 1$. A necessary condition for the envy cohesive strategy $S_{\mathbf{e}} = (Y,Y)$ to be envy Nash Equilibrium is*

$$\alpha_{12}^{N} + \alpha_{21}^{N} \geq \alpha_{11}^{N} + \alpha_{22}^{N}.$$

Proof. Using Theorem 3.7 , if $S_e = (Y,Y)$ is an *envy Nash Equilibrium* then one can arrange inequalities (3.28) and (3.30) to get

$$\begin{aligned} x &\geq -\alpha_{11}^{Y}(n_1 - 1) - \alpha_{12}^{Y} n_2 + \alpha_{22}^{Y}(n_2 - 1) + \alpha_{21}^{Y} n_1 + \alpha_{22}^{N} - \alpha_{21}^{N} + y, \\ y &\geq -\alpha_{22}^{Y}(n_2 - 1) - \alpha_{21}^{Y} n_1 + \alpha_{11}^{Y}(n_1 - 1) + \alpha_{12}^{Y} n_2 + \alpha_{11}^{N} + \alpha_{12}^{N} + x. \end{aligned}$$

The last two inequalities simplify to

$$\begin{aligned} x \geq\ & -\cancel{\alpha_{11}^{Y}(n_1 - 1)} - \cancel{\alpha_{12}^{Y} n_2} + \cancel{\alpha_{22}^{Y}(n_2 - 1)} \\ & + \cancel{\alpha_{21}^{Y} n_1} + \alpha_{22}^{N} - \alpha_{21}^{N} - \cancel{\alpha_{22}^{Y}(n_2 - 1)} \\ & - \cancel{\alpha_{21}^{Y} n_1} + \cancel{\alpha_{11}^{Y}(n_1 - 1)} + \cancel{\alpha_{12}^{Y} n_2} + \alpha_{11}^{N} + \alpha_{12}^{N} + x \end{aligned}$$

and

$$\begin{aligned} y \geq\ & -\cancel{\alpha_{22}^{Y}(n_2 - 1)} - \cancel{\alpha_{21}^{Y} n_1} + \cancel{\alpha_{11}^{Y}(n_1 - 1)} \\ & + \cancel{\alpha_{12}^{Y} n_2} + \alpha_{11}^{N} + \alpha_{12}^{N} - \cancel{\alpha_{11}^{Y}(n_1 - 1)} - \cancel{\alpha_{12}^{Y} n_2} \\ & + \cancel{\alpha_{22}^{Y}(n_2 - 1)} + \cancel{\alpha_{21}^{Y} n_1} + \alpha_{22}^{N} - \alpha_{21}^{N} + y. \end{aligned}$$

Simplifying the last two inequalities, we get

$$0 \geq \alpha_{22}^N - \alpha_{21}^N + \alpha_{11}^N - \alpha_{12}^N.$$

Hence,

$$\alpha_{12}^N + \alpha_{21}^N \geq \alpha_{11}^N + \alpha_{22}^N.$$

□

Hence, the last condition means that players of type t_i prefer more to be with players of type t_j deciding N rather being together deciding N. This is necessary condition for the strategy $S_{\mathbf{e}} = (Y, Y)$ to be *envy Nash Equilibrium*, where $i \neq j \in \{1, 2\}$. The condition agrees with the result of Lemma 3.2

Theorem 3.9. *Assume that* $\beta_1 = 1$ *and* $\beta_2 = 1$. *The envy cohesive strategy* $S_{\mathbf{e}} = (Y, N)$ *is an envy Nash Equilibrium if and only if* $(x, y) \in N_s^e(Y, N)$, *where the envy Nash domain* $N_s^e(Y, N)$ *is given by*

$$N_s^e(Y, N) = \{(x, y) \in \mathbb{R}^2 : y - x \leq \min\{Z_1^e(Y, N), Z_2^e(Y, N)\}\},$$

the envy threshold $Z_1^e(Y, N)$ *is given by*

$$Z_1^e(Y, N) = V(Y, N) - H(Y, N) + \alpha_{22}^Y + \alpha_{21}^N,$$

the envy threshold $Z_2^e(Y, N)$ *is given by*

$$Z_2^e(Y, N) = V(Y, N) - H(Y, N) + \alpha_{11}^N + \alpha_{12}^Y, \tag{3.32}$$

the horizontal $H(Y, N)$ *and vertical* $V(Y, N)$ *strategic thresholds of* (Y, N) *strategy are as given in* (2.20).

Proof. The cohesive envy strategy $S_{\mathbf{e}} = (Y, N)$ is *Nash Equilibrium* if and only if the following inequalities hold

$$\begin{aligned} U_1(Y; n_1, 0, 1) &\geq U_1(N; n_1 - 1, 0, 1) \quad \text{and} \\ U_2(N; n_1, 0, 1) &\geq U_2(Y; n_1, 1, 1). \end{aligned} \tag{3.33}$$

Substituting the envy utility functions from (3.23) and (3.24) in the first inequality of (3.33), we obtain

$$(\omega_1^Y - \omega_2^Y) + \alpha_{11}^Y(n_1 - 1) + \alpha_{12}^Y(0) - \alpha_{22}^Y(0 - 1)$$
$$-\alpha_{21}^Y n_1 \geq (\omega_1^N - \omega_2^N) + \alpha_{11}^N(n_1 - (n_1 - 1) - 1)$$
$$+\alpha_{12}^N(n_2 - 0) - \alpha_{22}^N(n_2 - 0 - 1) - \alpha_{21}^N(n_1 - (n_1 - 1)).$$

Rearrange the above inequality, we get

$$(\omega_1^Y - \omega_2^Y) - (\omega_1^N - \omega_2^N) \geq -\alpha_{22}^N(n_2 - 1)$$
$$-\alpha_{11}^Y(n_1 - 1) + \alpha_{12}^N n_2 + \alpha_{21}^Y n_1 - \alpha_{22}^Y - \alpha_{21}^N.$$

By simplifying the previous inequality using the definition of horizontal and vertical preferences x and y given in (2.8), we get

$$x - y \geq -\alpha_{22}^N(n_2 - 1) - \alpha_{11}^Y(n_1 - 1) + \alpha_{12}^N n_2$$
$$+\alpha_{21}^Y n_1 - \alpha_{22}^Y - \alpha_{21}^N. \tag{3.34}$$

Substituting the horizontal $H(Y, N)$ and vertical $V(Y, N)$ strategic thresholds from (2.20) in the last inequality, we obtain

$$y - x \leq V(Y, N) - H(Y, N) + \alpha_{22}^Y + \alpha_{21}^N.$$

Hence,

$$y - x \leq Z_1^e(Y, N),$$

where $Z_1^e(Y, N)$ is as given in the statement of Theorem 3.9.
Similarly, we substitute the envy utility functions from (3.26) and (3.25) in the second inequality of (3.33), we obtain

$$(\omega_2^N - \omega_1^N) + \alpha_{22}^N(n_2 - 0 - 1) + \alpha_{21}^N(n_1 - n_1)$$
$$-\alpha_{11}^N(n_1 - n_1 - 1) - \alpha_{12}^N(n_2 - 0) \geq (\omega_2^Y - \omega_1^Y)$$
$$+\alpha_{22}^Y(1 - 1) + \alpha_{21}^Y n_1 - \alpha_{11}^Y(n_1 - 1) - \alpha_{12}^Y(1).$$

By simplifying the previous inequality using the definition of horizontal and vertical preferences x and y given in (2.8), we get

$$x - y \geq -\alpha_{22}^N(n_2 - 1) - \alpha_{11}^Y(n_1 - 1) + \alpha_{21}^Y n_1$$
$$+\alpha_{12}^N n_2 - \alpha_{12}^Y - \alpha_{11}^N. \tag{3.35}$$

Substituting the horizontal $H(Y,N)$ and vertical $V(Y,N)$ strategic thresholds from (2.20) in the last inequality, we obtain

$$y - x \leq V(Y,N) - H(Y,N) + \alpha_{11}^N + \alpha_{12}^Y.$$

Hence,

$$y - x \leq Z_2^e(Y,N),$$

where $Z_2^e(Y,N)$ is as given in the statement of Theorem 3.9.

□

Theorem 3.10. *Assume that $\beta_1 = 1$ and $\beta_2 = 1$. The envy cohesive strategy $S_{\mathbf{e}} = (N,Y)$ is an envy Nash Equilibrium if and only if $(x,y) \in N_s^e(N,Y)$, where the envy Nash domain $N_s^e(N,Y)$ is given by*

$$N_s^e(N,Y) = \{(x,y) \in \mathbb{R}^2 : y - x \geq \max\{Z_1^e(N,Y), Z_2^e(N,Y)\}\},$$

the envy threshold $Z_1^e(N,Y)$ is given by

$$Z_1^e(N,Y) = V(N,Y) - H(N,Y) - \alpha_{21}^Y - \alpha_{22}^N,$$

the envy threshold $Z_2^e(N,Y)$ is given by

$$Z_2^e(N,Y) = V(N,Y) - H(N,Y) - \alpha_{11}^Y - \alpha_{12}^N, \tag{3.36}$$

the horizontal $H(N,Y)$ and vertical $V(N,Y)$ strategic thresholds of (N,Y) strategy are as given in (2.25).

Proof. The cohesive envy strategy $S_{\mathbf{e}} = (N,Y)$ is *Nash Equilibrium* if and only if the following inequalities hold

$$\begin{aligned} U_1(N;0,n_2,1) &\geq U_1(Y;1,n_2,1) \quad \text{and} \\ U_2(Y;0,n_2,1) &\geq U_2(N;0,n_2-1,1). \end{aligned} \tag{3.37}$$

Substituting the envy utility functions from (3.24) and (3.23) in the first inequality of (3.37), we obtain

$$\begin{aligned} &(\omega_1^N - \omega_2^N) + \alpha_{11}^N(n_1 - 0 - 1) + \alpha_{12}^N(n_2 - n_2) \\ &-\alpha_{22}^N(n_2 - n_2 - 1) - \alpha_{21}^N(n_1 - 0) \geq (\omega_1^Y - \omega_2^Y) \\ &+\alpha_{11}^Y(1 - 1) + \alpha_{12}^Y n_2 - \alpha_{22}^Y(n_2 - 1) - \alpha_{21}^Y(1). \end{aligned}$$

Rearrange the above inequality, we get

$$(\omega_2^Y - \omega_2^N) - (\omega_1^Y - \omega_1^N) \geq -\alpha_{22}^Y(n_2 - 1)$$
$$-\alpha_{11}^N(n_1 - 1) + \alpha_{21}^N n_1 + \alpha_{12}^Y n_2 - \alpha_{21}^Y - \alpha_{22}^N.$$

By simplifying the previous inequality using the definition of horizontal and vertical preferences x and y given in (2.8), we get

$$\begin{aligned} y - x \geq & -\alpha_{22}^Y(n_2 - 1) - \alpha_{11}^N(n_1 - 1) + \alpha_{21}^N n_1 \\ & +\alpha_{12}^Y n_2 - \alpha_{21}^Y - \alpha_{22}^N \end{aligned} \tag{3.38}$$

Substituting the horizontal $H(N, Y)$ and vertical $V(N, Y)$ strategic thresholds from (2.25) in the last inequality, we obtain

$$y - x \geq V(N, Y) - H(N, Y) - \alpha_{21}^Y - \alpha_{22}^N.$$

Hence,

$$y - x \geq Z_1^e(N, Y),$$

where $Z_1^e(N, Y)$ is as given in the statement of Theorem 3.10.
Similarly, we substitute the envy utility functions from (3.25) and (3.26) in the second inequality of (3.37), we obtain

$$\begin{aligned} &(\omega_2^Y - \omega_1^Y) + \alpha_{22}^Y(n_2 - 1) + \alpha_{21}^Y(0) \\ &-\alpha_{11}^Y(0 - 1) - \alpha_{12}^Y n_2 \geq (\omega_2^N - \omega_1^N) \\ &+\alpha_{22}^N(n_2 - (n_2 - 1) - 1) + \alpha_{21}^N(n_1 - 0) \\ &-\alpha_{11}^N(n_1 - 0 - 1) - \alpha_{12}^N(n_2 - (n_2 - 1)). \end{aligned}$$

Rearrange the above inequality, we get

$$\begin{aligned} \omega_2^Y - \omega_2^N - (\omega_1^Y - \omega_1^N) \geq & -\alpha_{11}^N(n_1 - 1) - \alpha_{22}^Y(n_2 - 1) \\ & +\alpha_{12}^Y n_2 + \alpha_{21}^N n_1 - \alpha_{11}^Y - \alpha_{12}^N. \end{aligned}$$

By simplifying the previous inequality using the definition of horizontal and vertical preferences x and y given in (2.8), we get

$$\begin{aligned} y - x \geq & -\alpha_{11}^N(n_1 - 1) - \alpha_{22}^Y(n_2 - 1) + \alpha_{12}^Y n_2 \\ & +\alpha_{21}^N n_1 - \alpha_{11}^Y - \alpha_{12}^N. \end{aligned} \tag{3.39}$$

Substituting *the horizontal* $H(N,Y)$ *and vertical* $V(N,Y)$ *strategic thresholds* from (2.25) in the last inequality, we obtain

$$y - x \geq V(N,Y) - H(N,Y) - \alpha_{11}^{Y} - \alpha_{12}^{N}.$$

Hence,

$$y - x \geq Z_2^e(N,Y),$$

where $Z_2^e(N,Y)$ is as given in the statement of Theorem 3.10. □

Theorem 3.11. *Assume that* $\beta_1 = 1$ *and* $\beta_2 = 1$*. The envy cohesive strategy* $S_{\mathbf{e}} = (N,N)$ *is an envy Nash Equilibrium if and only if* $(x,y) \in N_s^e(N,N)$*, where the envy Nash domain* $N_s^e(N,N)$ *is given by*

$$N_s^e(N,N) = \{(x,y) \in \mathbb{R}^2 : Z_L^e(N,N) \leq y - x \leq Z_R^e(N,N)\},$$

the left envy threshold $Z_L^e(N,N)$ *is given by*

$$Z_L^e(N,N) = H(N,N) - V(N,N) - \alpha_{22}^{Y} + \alpha_{21}^{Y},$$

the right envy threshold $Z_R^e(N,N)$ *is given by*

$$Z_R^e(N,N) = H(N,N) - V(N,N) + \alpha_{11}^{Y} - \alpha_{12}^{Y},$$

the horizontal $H(N,N)$ *and vertical* $V(N,N)$ *strategic thresholds of* (N,N) *strategy are as given in* (2.29)*.*

Proof. The cohesive envy strategy $S_{\mathbf{e}} = (N,N)$ is *Nash Equilibrium* if and only if the following inequalities hold

$$\begin{aligned} U_1(N;0,0,1) &\geq U_1(Y;1,0,1) \quad \text{and} \\ U_2(N;0,0,1) &\geq U_2(Y;0,1,1). \end{aligned} \tag{3.40}$$

Substituting the envy utility functions from (3.24) and (3.23) in the first inequality of (3.40), we obtain

$$\begin{aligned} &(\omega_1^N - \omega_2^N) + \alpha_{11}^N(n_1 - 0 - 1) + \alpha_{12}^N(n_2 - 0) \\ &-\alpha_{22}^N(n_2 - 0 - 1) - \alpha_{21}^N(n_1 - 0) \geq (\omega_1^Y - \omega_2^Y) \\ &\alpha_{11}^Y(1 - 1) + \alpha_{12}^Y(0) - \alpha_{22}^Y(0 - 1) - \alpha_{21}^Y(1). \end{aligned}$$

Rearrange the above inequality, we get

$$(\omega_1^N - \omega_2^N) - (\omega_1^Y - \omega_2^Y) \geq \alpha_{22}^Y - \alpha_{21}^Y + \alpha_{21}^N(n_1)$$
$$-\alpha_{11}^N(n_1 - 1) - \alpha_{12}^N(n_2) + \alpha_{22}^N(n_2 - 1).$$

By simplifying the previous inequality using the definition of horizontal and vertical preferences x and y given in (2.8), we get

$$\begin{aligned} y - x \geq \; & -\alpha_{11}^N(n_1 - 1) - \alpha_{12}^N(n_2) + \alpha_{22}^N(n_2 - 1) \\ & + \alpha_{21}^N(n_1) + \alpha_{22}^Y - \alpha_{21}^Y. \end{aligned} \tag{3.41}$$

Substituting the *horizontal* $H(N, N)$ *and vertical* $V(N, N)$ *strategic thresholds* from (2.29) in the last inequality, we obtain

$$y - x \geq H(N, N) - V(N, N) - \alpha_{22}^Y + \alpha_{21}^Y.$$

Hence,

$$x - y \geq Z_L^e(N, N), \tag{3.42}$$

where $Z_L^e(N, N)$ is as given in the statement of Theorem 3.11.
Similarly, we substitute the envy utility functions from (3.26) and (3.25) in the second inequality of (3.40), we obtain

$$(\omega_2^N - \omega_1^N) + \alpha_{22}^N(n_2 - 0 - 1) + \alpha_{21}^N(n_1 - 0)$$
$$-\alpha_{11}^N(n_1 - 0 - 1) - \alpha_{12}^N(n_2 - 0) \geq (\omega_2^Y - \omega_1^Y)$$
$$+\alpha_{22}^Y(1 - 1) + \alpha_{21}^Y(0) - \alpha_{11}^Y(0 - 1) - \alpha_{12}^Y(1).$$

Rearrange the above inequality, we get

$$(\omega_2^N - \omega_1^N) - (\omega_2^Y - \omega_1^Y) \geq \alpha_{11}^Y - \alpha_{12}^Y$$
$$-\alpha_{22}^N(n_2 - 1) - \alpha_{21}^N n_1 + \alpha_{11}^N(n_1 - 1) + \alpha_{12}^N n_2.$$

By simplifying the previous inequality using the definition of horizontal and vertical preferences x and y given in (2.8), we get

$$\begin{aligned} x - y \geq \; & -\alpha_{22}^N(n_2 - 1) - \alpha_{21}^N n_1 + \alpha_{11}^N(n_1 - 1) \\ & + \alpha_{12}^N n_2 + \alpha_{11}^Y - \alpha_{12}^Y. \end{aligned} \tag{3.43}$$

Substituting the *horizontal* $H(N,N)$ *and vertical* $V(N,N)$ *strategic thresholds* from (2.29) in the last inequality, we obtain

$$y - x \leq H(N,N) - V(N,N) + \alpha_{11}^{Y} - \alpha_{12}^{Y}.$$

Hence,

$$y - x \leq Z_R^e(N,N), \tag{3.44}$$

where $Z_R^e(N,N)$ is as given in the statement of Theorem 3.11. Combining inequalities (3.44) and (3.42), we get

$$Z_L^e(N,N) \leq y - x \leq Z_R^e(N,N).$$

□

Lemma 3.12. *Assume that* $\beta_1 = 1$ *and* $\beta_2 = 1$. *A necessary condition for the envy cohesive strategy* $S_{\mathbf{e}} = (N,N)$ *to be envy Nash Equilibrium is*

$$\alpha_{12}^{Y} + \alpha_{21}^{Y} \geq \alpha_{11}^{Y} + \alpha_{22}^{Y}.$$

Proof. Using Theorem 3.11 , if $S_e = (N,N)$ is an *envy Nash Equilibrium* then one can arrange inequalities (3.41) and (3.43) to get

$$\begin{aligned}
x \geq& \quad -\alpha_{22}^{N}(n_2-1) - \alpha_{21}^{N}n_1 + \alpha_{11}^{N}(n_1-1) + \alpha_{12}^{N}n_2 + \alpha_{11}^{Y} - \alpha_{12}^{Y} + y,\\
y \geq& \quad -\alpha_{11}^{N}(n_1-1) - \alpha_{12}^{N}n_2 + \alpha_{22}^{N}(n_2-1) + \alpha_{21}^{N}n_1 + \alpha_{22}^{Y} - \alpha_{21}^{Y} + x.
\end{aligned}$$

The last two inequalities simplify to

$$\begin{aligned}
x \quad \geq & \quad -\cancel{\alpha_{22}^{N}(n_2-1)} - \cancel{\alpha_{21}^{N}n_1} + \cancel{\alpha_{11}^{N}(n_1-1)} + \cancel{\alpha_{12}^{N}n_2}\\
& \quad +\alpha_{11}^{Y} - \alpha_{12}^{Y}, -\cancel{\alpha_{11}^{N}(n_1-1)} \cancel{-\alpha_{12}^{N}n_2} + \cancel{\alpha_{21}^{N}n_1}\\
& \quad +\cancel{\alpha_{22}^{N}(n_2-1)} + \alpha_{22}^{Y} - \alpha_{21}^{Y},
\end{aligned}$$

and

$$\begin{aligned}
y \quad \geq & \quad -\cancel{\alpha_{11}^{N}(n_1-1)} \cancel{-\alpha_{12}^{N}n_2} + \cancel{\alpha_{22}^{N}(n_2-1)} + \cancel{\alpha_{21}^{N}n_1}\\
& \quad +\alpha_{22}^{Y} - \alpha_{21}^{Y} - \cancel{\alpha_{22}^{N}(n_2-1)} - \cancel{\alpha_{21}^{N}n_1} + \cancel{\alpha_{11}^{N}(n_1-1)}\\
& \quad +\cancel{\alpha_{12}^{N}n_2} + \alpha_{11}^{Y} - \alpha_{12}^{Y}.
\end{aligned}$$

Simplifying the last two inequalities, we get

$$0 \geq \alpha_{22}^{Y} - \alpha_{21}^{Y} + \alpha_{11}^{Y} - \alpha_{12}^{Y}.$$

Hence,

$$\alpha_{21}^{Y} + \alpha_{12}^{Y} \geq \alpha_{22}^{Y} + \alpha_{11}^{Y}.$$

$\square$

Hence, the last condition means that players of type t_i prefer more to be with players of type t_j deciding Y rather being together deciding Y. This is necessary condition for the strategy $S_{\mathbf{e}} = (N, N)$ to be *envy Nash Equilibrium*, where $i \neq j \in \{1, 2\}$. The condition agrees with the result of Lemma 3.5.

Conclusions

We have presented an envy behavioral game theoretical model for two homogeneous types of players. As well as, we have characterized all envy strategies that form Nash equilibria and determined the corresponding envy Nash domains for each type of players.And we have compared between the Nash domains and the envy Nash domains. Then, we have studied the geometric envy tilings and showed that there are 1024 combinatorial classes of envy decision tilings, 256 of them are being structurally stable while 768 have bifurcation. Moreover, we have stated some conditions for which the disparate envy strategic set is a Nash Equilibrium. Finally, we have studied the special case for Theorem 3.1 when the envy parameters $\beta_1 = 1$ and $\beta_2 = 1$.

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